MINORS OF GRAPHS OF LARGE PATH-WIDTH

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To my parents.

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SUMMARY

Let P be a graph with a vertex v such that $P - v$ is a forest and let Q be an outerplanar graph. In 1993 Paul Seymour asked if every two-connected graph of sufficiently large path-width contains P or Q as a minor. Define $g(H)$ as the minimum number for which there exists a positive integer $p(H)$ such that every $g(H)$-connected H-minor-free graph has path-width at most $p(H)$. Then $g(H) = 0$ iff H is a forest and there is no graph H with $g(H) = 1$, because path-width of a graph G is the maximum of the path-widths of its connected components. Let A be the graph that consists of a cycle $(a_1, a_2, a_3, a_4, a_5, a_6, a_1)$ and extra edges a_1a_3, a_3a_5, a_5a_1. Let $C_{3,2}$ be a graph of 2 disjoint triangles. In 2014 Marshall and Wood conjectured that a graph H does not have K_4, $K_{2,3}$, $C_{3,2}$ or A as a minor if and only if $g(H) \leq 2$. In this thesis we answer Paul Seymour's question in the affirmative and prove Marshall and Wood's conjecture, as well as extend the result to three-connected and four-connected graphs of large path-width. We introduce "cascades", our main tool, and prove that in any tree-decomposition with no duplicate bags of bounded width of a graph of big path-width there is an "injective" cascade of large height. Then we prove that every 2-connected graph of big path-width and bounded tree-width admits a tree-decomposition of bounded width and a cascade with linkages that are minimal. We analyze those minimal linkages and prove that there are essentially only two types of linkage. Then we convert the two types of linkage into the two families of graphs P and Q. In this process we have to choose the "right" tree decomposition to deal with special cases like a long cycle. Similar techniques are used for three-connected and four-connected graphs with high path-width.

CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 Introduction

All *graphs* in this thesis are finite and simple; that is, they have no loops or parallel edges. *Paths* and *cycles* have no "repeated" vertices or edges. A graph H is a *minor* of a graph G if we can obtain H by contracting edges of a subgraph of G. An H *minor* is a minor isomorphic to H. A tree-decomposition of a graph G is a pair (T, X), where T is a tree and X is a family $(X_t : t \in V(T))$ such that:

(W1) $\bigcup_{t \in V((T)} X_t = V(G)$, and for every edge of G with ends u and v there exists $t \in V(T)$ such that $u, v \in X_t$, and

(W2) if $t_1, t_2, t_3 \in V(T)$ and t_2 lies on the path in T between t_1 and t_3, then $X_{t_1} \cap X_{t_3} \subseteq X_{t_2}$.

The *width* of a tree-decomposition (T, X) is $\max\{|X_t| - 1 : t \in V(T)\}$. The *tree-width* of a graph G is the smallest width among all tree-decompositions of G. A *path-decomposition* of G is a tree-decomposition (T, X) of G, where T is a path. We will often denote a path-decomposition as $(X_1, X_2, \ldots, X_n)$, rather than having the constituent sets indexed by the vertices of a path. The *path-width* of G is the smallest width among all path-decompositions of G. The concept of tree-width and path-width is useful in structural graph theory [19, 22, 23, 24, 25, 27], as well as in theory of algorithms and computation [1, 2, 8, 7, 26]. Robertson and Seymour [21] proved the following:

Theorem 1.1.1. *For every planar graph H there exists an integer $n = n(H)$ such that every graph of tree-width at least n has an H minor.*

Robertson and Seymour [20] also proved an analogous result for path-width:

Theorem 1.1.2. *For every forest F, there exists an integer $p = p(F)$ such that every graph of path-width at least p has an F minor.*

Bienstock, Robertson, Seymour and Thomas [3] gave a simpler proof of Theorem 1.1.2 and improved the value of p to $|V(F)| - 1$, which is best possible, because K_k has path-width $k - 1$ and does not have any forest minor on $k + 1$ vertices. A yet simpler proof of Theorem 1.1.2 was found by Diestel [12].

While Geelen, Gerards and Whittle [16] generalized Theorem 1.1.1 to representable matroids, it is not *a priori* clear what a version of Theorem 1.1.2 for matroids should be, because excluding a forest in matroid setting is equivalent to imposing a bound on the number of elements and has no relevance to path-width. To overcome this, Seymour [11, Open Problem 2.1] asked if there was a generalization of Theorem 1.1.2 for 2-connected graphs with forests replaced by two families of graphs. In [9] we answer Seymour's question in the affirmative:

Theorem 1.1.3. *Let P be a graph with a vertex v such that $P \backslash v$ is a forest, and let Q be an outerplanar graph. Then there exists an integer $p = p(P, Q)$ such that every 2-connected graph of path-width at least p has a P or Q minor.*

Theorem 1.1.3 is a generalization of Theorem 1.1.2. To deduce Theorem 1.1.2 from Theorem 1.1.3, given a graph G, we may assume that G is connected, because the path-width of a graph is equal to the maximum path-width of its components. We add one vertex and make it adjacent to every vertex of G. Then the new graph is 2-connected, and by Theorem 1.1.3, it has a P or Q minor. By choosing suitable P and Q, we can get an F minor in G.

Marshall and Wood [17] define $g(H)$ as the minimum number for which there exists a positive integer $p(H)$ such that every $g(H)$-connected graph with no H minor has path-width at most $p(H)$. Then Theorem 1.1.2 implies that $g(H) = 0$ iff H is a forest. There is no graph H with $g(H) = 1$, because path-width of a graph G is the maximum of the path-widths of its connected components. Let A be the graph that consists of a cycle $a_1a_2a_3a_4a_5a_6a_1$ and extra edges a_1a_3, a_3a_5, a_5a_1. Let $C_{3,2}$ be the graph consisting of two disjoint triangles. We prove the following, conjectured by Marshall and Wood [17]:

Theorem 1.1.4. *A graph H has no $K_4, K_{2,3}$, $C_{3,2}$ or A minor if and only if $g(H) \leq 2$.*

Let P' be a graph with two distinct vertices u_1, u_2 such that $P' \backslash \{u_1, u_2\}$ is a forest, Q' be a graph with a vertex v such that $Q' \backslash \{v\}$ is an outerplanar graph, and R' be a tree with a cycle going through its leaves in order from the leftmost leaf to the rightmost leaf so that R' is planar. Then Theorem 1.1.3 can be generalized to 3-connected graphs of high path-width as follows.

Theorem 1.1.5. *There exists an integer $p = p(P', Q', R')$ such that every 3-connected graph of path-width at least p has a P', Q' or R' minor.*

Let P'' be a graph with three distinct vertices u_1, u_2, u_3 such that $P'' \backslash \{u_1, u_2, u_3\}$ is a forest, Q'' be a graph with two distinct vertices w_1, w_2 such that $Q'' \backslash \{w_1, w_2\}$ is an outerplanar graph, R'' be R' plus a vertex v such that v is adjacent to leaves of the tree in R', and S'' be a planar graph that consists of an outerplanar graph with a cycle going through its degree-2 vertices. Then Theorem 1.1.3 can also be generalized to 4-connected graphs of high path-width as follows.

Theorem 1.1.6. *There exists an integer $p = p(P'', Q'', R'', S'')$ such that every 4-connected graph of path-width at least p has a P'', Q'', R'' or S'' minor.*

Theorem 1.1.6 implies Theorem 1.1.5, which in turn implies Theorem 1.1.3, but we are presenting these results separately, because most of the lemmas from the lower connectivity cases are needed for the cases of higher connectivity.

The rest of the thesis is organized as follows. In the next sections, we introduce several basic concepts and terminologies needed for the following chapters, prove that the families of the graphs in the theorems above are necessary, and discuss several related conjectures.

In Chapter 2, we present our result in [10] that if a graph has a tree-decomposition of width at most w, then it has a special tree-decomposition of width at most w with certain desirable properties in Theorem 2.1.4. In Chapter 3, we prove Conjecture 1.1.4 follows from Theorem 1.1.3 and use the special tree-decomposition in Chapter 2 to prove Theorem 1.1.3. In Chapter 4 and Chapter 5, we use the same special tree-decomposition in Chapter 2 to prove Theorem 1.1.5 and Theorem 1.1.6.

1.2 Basic concepts and terminology

Definition Let $h \geq 0$ be an integer. By a *binary tree of height* h we mean a tree with a unique vertex r of degree two and all other vertices of degree one or three such that every vertex of degree one is at distance exactly h from r. Such a tree is unique up to isomorphism and so we will speak of the binary tree of height h. We denote the binary tree of height h by CT_h and we call r the *root* of CT_h. Each vertex in CT_h with distance k from r has *height* k. We call the vertices at distance h from r the *leaves of* CT_h. If t belongs to the unique path in CT_h from r to a vertex $t' \in V(T_h)$, then we say that t' is a *descendant* of t and that t is an *ancestor* of t'. If, moreover, t and t' are adjacent, then we say that t is the *parent* of t' and that t' is a *child* of t.

Let $\mathcal{P}_k$ be the graph consisting of CT_k and a separate vertex that is adjacent to every leaf of CT_k.

Definition Let $\mathcal{Q}_1$ be K_3. An arbitrary edge of $\mathcal{Q}_1$ will be designated as *base edge*. The remaining vertex is called *leaf*. For $i \geq 2$ the graph $\mathcal{Q}_i$ is constructed as follows: Now assume that $\mathcal{Q}_{i-1}$ has already been defined, and let Q_1 and Q_2 be two disjoint copies of $\mathcal{Q}_{i-1}$ with base edges u_1v_1 and u_2v_2, respectively. Let T be a copy of K_3 with vertex-set $\{w_1, w_2, w\}$ disjoint from Q_1 and Q_2. The graph $\mathcal{Q}_i$ is obtained from $Q_1 \cup Q_2 \cup T$ by identifying u_1 with w_1, u_2 with w_2, and v_1 and v_2 with w. The edge w_1w_2 will be the *base edge* of $\mathcal{Q}_i$, and the leaves of Q_1 or Q_2 will be the *leaves* of $\mathcal{Q}_i$.

A graph is *outerplanar* if it has a drawing in the plane (without crossings) such that every vertex is incident with the unbounded face. A graph is a *near-triangulation* if it is drawn in the plane in such a way that every face except possibly the unbounded one is bounded by a triangle.

Let H and G be graphs. If G has an H minor, then to every vertex u of H there corresponds a connected subgraph of G, called the *node of* u.

Definition For any positive integer k, let T be CT_{k+1} after contracting its root and two children of the root to one vertex. let $\mathcal{P}'_k$ be T plus two distinct vertices each adjacent to the leaves of T. Let $\mathcal{Q}'_k$ be $\mathcal{Q}_k$ plus a vertex adjacent to its leaves. Let $\mathcal{R}'_k$ be T plus a cycle going through its leaves in order from the leftmost leaf to the right most leaf.

Definition By a *ternary tree of height* h we mean a tree whose vertices have degree one or four such that there exists a vertex r of degree four such that every vertex of degree one is at distance exactly h from a vertex r. For any positive integer k, let $\mathcal{P}''_k$ be a ternary tree of height k plus three distinct vertices each adjacent to the leaves of the tree. Let $\mathcal{Q}''_k$ be $\mathcal{Q}_k$ plus two distinct vertices adjacent to its leaves. Let $\mathcal{R}''_k$ be a ternary tree of height k plus a cycle going through its leaves in order from the leaftmost leaf to the rightmost leaf and a vertex adjacent to every leaf of the ternary tree. Let $\mathcal{S}''_k$ be a planar graph consisting of $\mathcal{Q}_k$ and a cycle going through its leaves.

Let T, T' be trees. A *homeomorphic embedding of* T *into* T' is a mapping $\eta : V(T) \to V(T')$ such that

- η is an injection, and
- if tt_1, tt_2 are edges of T with a common end, and P_i is the unique path in T' with ends $\eta(t)$ and $\eta(t_i)$, then P_1 and P_2 are edge-disjoint.

We will write $\eta : T \hookrightarrow T'$ to denote that η is a homeomorphic embedding of T into T'.

For every integer $h \geq 1$ we will need a specific type of tree, which we will denote by T_h. The tree T_h is obtained from CT_h by subdividing every edge not incident with a vertex of degree one exactly once, and adding a new vertex r' of degree one adjacent to the root r of CT_h. The vertices of T_h of degree three will be called *major*, and all the other vertices will be called *minor*. We say that r is the *major root* of T_h and that r' is the *minor root* of T_h. Each major vertex at distance $2k$ from r has *height* k, and each minor vertex at distance $2k$ from r' has *height* k.

If t belongs to the unique path in T_h from r' to a vertex $t' \in V(T_h)$, then we say that t' is a *descendant* of t and that t is an *ancestor* of t'. If, moreover, t and t' are adjacent, then we say that t is the *parent* of t' and that t' is a *child* of t. Thus every major vertex t has exactly three minor neighbors. Exactly one of those neighbors is an ancestor of t. The other two neighbors are descendants of t. We will assume that one of the two descendant neighbors is designated as the *left neighbor* and the other as the *right neighbor*. Let t_0, t_1, t_2 be the parent, left neighbor and right neighbor of t, respectively. We say that the ordered triple (t_0, t_1, t_2) is the *trinity at* t. In case we want to emphasize that the trinity is at t, we use the notation $(t_0(t), t_1(t), t_2(t))$.

Let h, h' be integers. We say that a homeomorphic embedding $\gamma : T_{h'} \hookrightarrow T_h$ is *monotone* if

- t is a major vertex of $T_{h'}$ with trinity (t_1, t_2, t_3), then $\gamma(t_2)$ is the left neighbor of $\gamma(t)$ and $\gamma(t_3)$ is the right neighbor of $\gamma(t)$, and
- the image under γ of the minor root of $T_{h'}$ is the minor root of T_h.

Let G be a graph, let $v \in V(G)$ and for $i = 1, 2, 3$ let P_i be a path in G with ends v and v_i such that the paths P_1, P_2, P_3 are pairwise disjoint, except for v. Assume that at least two of the paths P_i have length at least one. We say that $P_1 \cup P_2 \cup P_3$ is a *tripod* with *center* v and *feet* v_1, v_2, v_3.

Let T be a tree. If $t_1, t_2 \in V(T)$, then by $t_1 T t_2$ we denote the unique path in T with ends t_1 and t_2 and by $T[t_1, t_2]$ we denote the vertex-set of $t_1 T t_2$.

1.3 The necessity of the families

The families of graphs in Theorem 1.1.3, Theorem 1.1.5, and Theorem 1.1.6 are necessary. To show this, for each $R \in \{P, Q, P', Q', R', P'', Q'', R'', S''\}$, let F_R be the set of minors of the graphs having form R. We need to show that $F_{R_1} \not\subseteq F_{R_2}$ for all distinct $R_1, R_2 \in \{P, Q\}$, $F_{R_1} \not\subseteq F_{R_2}$ for all distinct $R_1, R_2 \in \{P', Q', R'\}$, and $F_{R_1} \not\subseteq F_{R_2}$ for all distinct $R_1, R_2 \in \{P'', Q'', R'', S''\}$.

For Theorem 1.1.3, $K_{2,3} \in F_P$ but $K_{2,3} \notin F_Q$ because Q is outerplanar. Also, $C_{3,2} \in F_Q$ but $C_{3,2} \notin F_P$ because every cycle in P shares a common vertex. Therefore we need both families of graphs P and Q.

For Theorem 1.1.5, there are two vertices $u, v \in V(P')$ such that for every cycle C in P', $V(C) \cap \{u, v\} \neq \emptyset$, and there are some Q' containing three disjoint cycles, so $F_{Q'} \not\subseteq F_{P'}$. The graph Q' is always planar but $K_{3,3} \in F_{P'}$, so $F_{P'} \not\subseteq F_{Q'}$. Similarly, R' is planar so $F_{P'} \not\subseteq F_{R'}$, and some R' has three disjoint cycles as a subgraph, so $F_{R'} \not\subseteq F_{P'}$. There is no vertex in $\mathcal{R}'_4$ whose removal makes the remaining graph outerplanar, but there is such a vertex in every Q', so $\mathcal{R}'_4 \notin F_{Q'}$ and therefore, $F_{R'} \not\subseteq F_{Q'}$. We will show that $\mathcal{Q}'_4$ is not a minor of any R'. In fact, assume there exists R' such that $\mathcal{Q}'_4$ is a minor of R'. For every $u \in V(\mathcal{Q}'_4)$, denote the vertex set of the node of u in R' as $N(u)$. Let the base edge of $\mathcal{Q}'_4$ be v_1v_2 and $v \in V(\mathcal{Q}'_4)$ be the only vertex adjacent to both v_1 and v_2. Let $v_1v_3v_4, v_2v_5v_6$ be the two K_3 in $\mathcal{Q}'_4$ such that v_4, v_6 are leaves of the $\mathcal{Q}_4$ in $\mathcal{Q}'_4$. Let $v_7v_8v_9$ be another K_3 in $\mathcal{Q}'_4$ such that $\{v_7, v_8, v_9\}$ is disjoint from $\{v, v_1, v_2, v_3, v_4, v_5, v_6\}$ and v_9 is a leaf of the $\mathcal{Q}_4$ in $\mathcal{Q}'_4$. Let $r \in V(\mathcal{Q}'_4)$ be the vertex adjacent to the leaves of the $\mathcal{Q}_4$ in $\mathcal{Q}'_4$. Let T and C be the tree and the cycle in R' as in the definition of R'. Because $R' \backslash E(C)$ is acyclic, there exist $a_1 \in N(v_1) \cup N(v_3) \cup N(v_4)$, $a_2 \in N(v_2) \cup N(v_5) \cup N(v_6)$, and $a_3 \in N(v_7) \cup N(v_8) \cup N(v_9)$ such that $a_1, a_2, a_3 \in V(C)$. Also, there exist $b \in N(v)$ and $c \in N(r)$ and paths $P_1, P_2, P_3, Q_1, Q_2, Q_3$ in R' such that P_i is from a_i to b for all $i \in \{1, 2, 3\}$, Q_i is from a_i to c for all $i \in \{1, 2, 3\}$, P_1, P_2, P_3 are internally disjoint,

Q_1, Q_2, Q_3 are internally disjoint, and P_i and Q_j are disjoint for all distinct $i, j \in \{1, 2, 3\}$. However, this is impossible because when drawn in a plane, $R' \backslash E(C)$ is planar and lies only in one of the two faces created by C. Hence, $\mathcal{Q}_4' \notin F_{R'}$ and that means $F_{Q'} \nsubseteq F_{R'}$.

For Theorem 1.1.6, there are three vertices $u, v, w \in V(P'')$ such that for every cycle C in P'', $V(C) \cap \{u, v, w\} \neq \emptyset$, and there are some Q'', R'', S'' containing four disjoint cycles as a subgraph, so $F_{Q''} \nsubseteq F_{P''}$, $F_{R''} \nsubseteq F_{P''}$, and $F_{S''} \nsubseteq F_{P''}$. The graph $K_{3,3}$ is a minor of some P'' but R'' and S'' are planar, so $F_{P''} \nsubseteq F_{R''}$ and $F_{P''} \nsubseteq F_{S''}$. Similarly, $K_{3,3}$ is a minor of some Q'', so $F_{Q''} \nsubseteq F_{R''}$ and $F_{Q''} \nsubseteq F_{S''}$. The graph $K_{4,4}$ is a minor of some P'' but is not a minor of any Q'', so $F_{P''} \nsubseteq F_{Q''}$. There are no two vertices in $\mathcal{R}_4''$ whose removal makes the remaining graph outerplanar, but there are such two vertices in every Q'', so $\mathcal{R}_4'' \notin F_{Q''}$ and therefore, $F_{R''} \nsubseteq F_{Q''}$. Similarly, there are no two vertices in $\mathcal{S}_3''$ whose removal makes the remaining graph outerplanar, but there are such two vertices in every Q'', so $\mathcal{S}_3'' \notin F_{Q''}$ and therefore, $F_{S''} \nsubseteq F_{Q''}$. The graph $\mathcal{S}_7''$ is not a minor of any R''. In fact, assume $\mathcal{S}_7''$ is a minor of some R''. Let $v \in V(R'')$ be the vertex that is adjacent to every leaf of the tree in R''. Let H be the disjoint union of two $\mathcal{Q}_4'$, then H is a minor of $\mathcal{S}_7''$, so H is a minor of R''. This implies $\mathcal{Q}_4'$ is a minor of $R'' \backslash v$, which has form R'. But from above $\mathcal{Q}_4'$ is not a minor of any R', so this is a contradiction. Hence, $\mathcal{S}_7'' \notin F_{R''}$ and that means $F_{S''} \nsubseteq F_{R''}$. The graph $\mathcal{R}_3''$ is not a minor of any S''. In fact, assume $\mathcal{R}_3''$ is a minor of some S''. Let C be the cycle as in the definition of S''. Let u be the root of the tree in $\mathcal{R}_3''$ and v be the vertex that is adjacent to every leaf of the tree in $\mathcal{R}_3''$. Let H_1, H_2, H_3 be three graphs isomorphic to K_4 and let H be the disjoint union of H_1, H_2, H_3. Then $\mathcal{R}_3'' \backslash \{u, v\}$ has an H minor. For $x \in V(\mathcal{R}_3'')$, let $N(x)$ be the vertex set of the node of x in S''. Because $S'' \backslash E(C)$ is outerplanar, there exist $r_i \in V(H_i)$ and $a_i \in N(r_i)$ for all $i \in \{1, 2, 3\}$ such that $a_i \in V(C)$ for all $i \in \{1, 2, 3\}$. Also, there exist $b \in N(u)$ and $c \in N(v)$ and paths $P_1, P_2, P_3, Q_1, Q_2, Q_3$ such that P_i is from a_i to b for all $i \in \{1, 2, 3\}$, Q_i is from a_i to c for all $i \in \{1, 2, 3\}$, P_1, P_2, P_3 are internally disjoint, Q_1, Q_2, Q_3 are internally disjoint, and P_i and Q_j are disjoint for distinct $i, j \in \{1, 2, 3\}$. However, this is impossible because

$S''\backslash E(C)$ is planar and lies only in one of the two faces created by C. Hence, $\mathcal{R}_3'' \notin F_{S''}$ and that means $F_{R''} \not\subseteq F_{S''}$.

1.4 Discussion

A family $\mathcal{F}$ of graphs is closed under minors if for every graph $G \in \mathcal{F}$, every graph isomorphic to a minor of G is also in $\mathcal{F}$. A family of graphs $\mathcal{H}$ is called a list of excluded minors of a family $\mathcal{F}$ of graphs if for every graph G we have that $G \in \mathcal{F}$ if and only if G does not have a minor isomorphic to any graph in $\mathcal{H}$. It is easy to see that every family of graphs that is closed under minors can be characterized by a list of excluded minors. The Graph Minor Theorem of Robertson and Seymour [27] states that this list is finite.

Theorem 1.4.1. *Every family of graphs that is closed under minors can be characterized by a finite list of excluded minors.*

Define $\mathcal{T}_l = \{H : g(H) \leq l\}$, where the function g is defined prior to Theorem 1.1.4. It is easy to see that $\mathcal{T}_l$ is closed under minors, so by the Graph Minor Theorem, we can characterize these classes of graphs by a finite list of excluded minors. From Theorem 1.1.2, the only excluded minor of $\mathcal{T}_0$ and $\mathcal{T}_1$ is K_3. From Theorem 1.1.4, the list of excluded minors for $\mathcal{T}_2$ is $\{K_4, K_{2,3}, C_{3,2}, A\}$. We can also characterize $\mathcal{T}_3$ and $\mathcal{T}_4$ by excluded minors. For $\mathcal{T}_3$, the list of these minors will be the union of the three lists of excluded minors of $F_{P'}$, $F_{Q'}$, and $F_{R'}$. The excluded minors of the graphs in $F_{P'}$ are the excluded minors of the class of graphs with minimum feedback vertex set size at most 2. In [6] there is a complete list of these minors. The graphs in $F_{Q'}$ can be characterized by the minors that do not have a vertex whose removal makes them outerplanar. We have found over 35 of such minors. The list of excluded minors of $F_{R'}$ is not known yet.

One related question is the general problem of Theorem 1.1.3 for a-connected graphs, where $a \geq 5$. For this, we are interested in finding a list of families of graphs that are a-connected and have large path-width when k is large, like $(\mathcal{P}_k, \mathcal{Q}_k)$ in Theorem 3.5.1. If we know this, then we can determine $H \in \mathcal{T}_a$ or not for any graph H. As shown above, the families of graphs for the cases $a \leq 4$ are based on the binary tree and the outerplanar graph, but this might be not true for general a. Also, there is no planar graph in the list when $a \geq 6$, so we cannot use Theorem 1.1.1 to limit the tree-width of the graph G anymore. For this, one can expect to use other techniques such as in [4] to deal with graphs of large tree-width.

The authors in [5] proved that for any positive integers k, w, a ($a \geq 3$), there exist $n = n(k, w)$ and $c = c(a)$ such that every c-connected graph G of tree-width at most w and of order at least n contains $K_{a,k}$ as a minor. Their proof used $c(a) = 2a + 1$ for $a = 3$, and $c(a) = 264a + 1$ for $a \geq 4$. The first part of their proof was to find a homeomorphic embedding of a path into tree T of a chosen tree decomposition (T, X) of G, such that the underlying subgraph of G formed by the union of bags on this path satisfies some special conditions. This is similar to what we did in our proof of Theorem 1.1.3, but we relied on a path-decomposition and can get a binary tree instead of a path. Therefore, we can get more structure of the underlying subgraph. The authors of [5] asked if $c(a)$ can be reduced to $2a + 1$:

Conjecture 1. *For any positive integers a, k, there exists an integer $N(a, k)$ such that every $(2a + 1)$-connected graph G on at least $N(a, k)$ vertices contains $K_{a,k}$ as a minor.*

A similar problem but with extra conditions on the tree-width and path-width might be related to our results:

Conjecture 2. *For any positive integers k, w, a ($a \geq 4$), there exists $p = p(k, w, a)$ such that every $(2a + 1)$-connected graph of tree-width at most w and path-width at least p contains $K_{a,k}$ as a minor.*

The conditions will help us get a homeomorphic embedding as in our proof for Theorem 1.1.3. If we can handle Conjecture 2, again we can use the techniques as in [4] to deal with graphs of large tree-width or small path-width to tackle Conjecture 1. The number

$c(a)$ cannot be smaller than $2a+1$ in Conjecture 1, because in [5] the authors show an infinite sequence of $2a$-connected graphs without a $K_{a,2a+1}$-minor. For Conjecture 2, it is not known $c(a)$ can be smaller than $2a+1$ or not. If Conjecture 2 is also true when we remove the tree-width condition, then $g(K_{a,k}) \leq 2a+1$. We have $g(K_{1,k}) = 0$ because $K_{1,k}$ is a tree. From a result of Ding [14], we can imply $g(K_{2,k}) = 3$. So another related question is whether $2a+1$ is the right bound for $g(K_{a,k})$? We made the following conjecture:

Conjecture 3. *For every integers $a \geq 1$ and $k \geq a$, $g(K_{a,k}) = a+1$.*

We have $g(K_3) = 2$. From above and a result of Dirac [15], we have $g(K_4) = 3$. It is known [18] that every large enough $(a+1)$-connected graph has a K_a minor, so $g(K_a) \leq a+1$. For $a \geq 5$, we can also construct a large path-width a-connected graph such that it has $a-5$ vertices whose removal make it planar. This means $g(K_a) > a$ for $a \geq 5$. So we have $g(K_a) = a+1$ for every $a \geq 5$. Define $\tau(H)$ as the size of a minimum feedback vertex set of the graph H. Then it is easy to see that $g(H) \geq \tau(H)+1$. Then we are also interested in the following question:

Open problem 1. *Is there an upper bound on $g(H)$ that is a function of $\tau(H)$?*

CHAPTER 2

A SPECIAL TREE DECOMPOSITION

The chapter is organized as follows. In the next section we review known results about tree-decompositions and state the main result of this chapter, Theorem 2.1.4. In Section 2.2 we introduce a linear quasi-order on the class of finite trees and prove a key lemma—Lemma 2.2.5. In Section 2.3 we prove Theorem 2.1.4, which we restate as Theorem 2.3.8.

2.1 Linked tree-decompositions

In this section we review properties of tree-decompositions established in [19, 28], and state the main result of this chapter. The proof of the following easy lemma can be found, for instance, in [28].

Lemma 2.1.1. *Let (T, Y) be a tree-decomposition of a graph G, and let H be a connected subgraph of G such that $V(H) \cap Y_{t_1} \neq \emptyset \neq V(H) \cap Y_{t_2}$, where $t_1, t_2 \in V(T)$. Then $V(H) \cap Y_t \neq \emptyset$ for every $t \in V(T)$ on the path between t_1 and t_2 in T.*

A tree-decomposition (T, Y) of a graph G is said to be *linked* if

(W3) for every two vertices t_1, t_2 of T and every positive integer k, either there are k disjoint paths in G between Y_{t_1} and Y_{t_2}, or there is a vertex t of T on the path between t_1 and t_2 such that $|Y_t| < k$.

It is worth noting that, by Lemma 2.1.1, the two alternatives in (W3) are mutually exclusive. The following is proved in [28].

Lemma 2.1.2. *If a graph G admits a tree-decomposition of width at most w, where w is some integer, then G admits a linked tree-decomposition of width at most w.*

Let (T, Y) be a tree-decomposition of a graph G, let $t_0 \in V(T)$, and let B be a component of $T \backslash t_0$. We say that a vertex $v \in Y_{t_0}$ is B-*tied* if $v \in Y_t$ for some $t \in V(B)$. We say that a path P in G is B-*confined* if $|V(P)| \geq 3$ and every internal vertex of P belongs to $\bigcup_{t \in V(B)} Y_t - Y_{t_0}$. We wish to consider the following three properties of (T, Y):

(W4) if t, t' are distinct vertices of T, then $Y_t \neq Y_{t'}$,

(W5) if $t_0 \in V(T)$ and B is a component of $T \backslash t_0$, then $\bigcup_{t \in V(B)} Y_t - Y_{t_0} \neq \emptyset$,

(W6) if $t_0 \in V(T)$, B is a component of $T \backslash t_0$, and u, v are B-tied vertices in Y_{t_0}, then there is a B-confined path in G between u and v.

The following strengthening of Lemma 2.1.2 is proved in [19].

Lemma 2.1.3. *If a graph G has a tree-decomposition of width at most w, where w is some integer, then it has a tree-decomposition of width at most w satisfying* (W1)–(W6).

We need one more condition, which we now introduce. Let T be a tree. A *triad in T* is a triple t_1, t_2, t_3 of vertices of T such that there exists a vertex t of T, called the *center*, such that t_1, t_2, t_3 belong to different components of $T \backslash t$. Let (T, W) be a tree-decomposition of a graph G, and let t_1, t_2, t_3 be a triad in T with center t_0. The *torso of* (T, W) *at* t_1, t_2, t_3 is the subgraph of G induced by the set $\bigcup W_t$, the union taken over all vertices $t \in V(T)$ such that either $t \in \{t_1, t_2, t_3\}$, or for all $i \in \{1, 2, 3\}$, the vertex t belongs to the component of $T \backslash t_i$ containing t_0. We say that the triad t_1, t_2, t_3 is W-*separable* if, letting $X = W_{t_1} \cap W_{t_2} \cap W_{t_3}$, the graph obtained from the torso of (T, W) at t_1, t_2, t_3 by deleting X can be partitioned into three disjoint non-null graphs H_1, H_2, H_3 in such a way that for all distinct $i, j \in \{1, 2, 3\}$ and all $t \in T[t_j, t_0]$, $|V(H_i) \cap W_t| \geq |V(H_i) \cap W_{t_j}| = |W_{t_j} - X|/2 \geq 1$. (Let us remark that this condition implies that $|W_{t_1}| = |W_{t_2}| = |W_{t_3}|$ and $V(H_i) \cap W_{t_i} = \emptyset$ for $i = 1, 2, 3$.) The last property of a tree-decomposition (T, W) that we wish to consider is

(W7) if t_1, t_2, t_3 is a W–separable triad in T with center t, then there exists an integer $i \in \{1, 2, 3\}$ with $W_{t_i} \cap W_t - (W_{t_1} \cap W_{t_2} \cap W_{t_3}) \neq \emptyset$.

Recall that conditions (W1) and (W2) were introduced at the beginning of Chapter 1, condition (W3) was introduced at the beginning of Chapter 2, and conditions (W4)–(W6) were introduced prior to Lemma 2.1.3. The following is the main result of this chapter.

Theorem 2.1.4. *If a graph G has a tree-decomposition of width at most w, where w is some integer, then it has a tree-decomposition of width at most w satisfying* (W1)–(W7).

2.2 A Quasi-order on trees

A *quasi-ordered set* is a pair $(Q, \leq)$, where Q is a set and $\leq$ is a *quasi-order*; that is, a reflexive and transitive relation on Q. If $q, q' \in Q$ we define $q < q'$ to mean that $q \leq q'$ and $q' \not\leq q$. We say that q, q' are $\leq$*-equivalent* if $q \leq q' \leq q$. We say that $(Q, \leq)$ is a *linear quasi-order* if for every two elements $q, q' \in Q$ either $q \leq q'$ or $q' \leq q$ or both. Let $(Q, \leq)$ be a linear quasi-order. If $A, B \subseteq Q$ we say that B $\leq$*-dominates* A if the elements of A can be listed as $a_1 \geq a_2 \geq \cdots \geq a_k$ and the elements of B can be listed as $b_1 \geq b_2 \geq \cdots \geq b_l$, and there exists an integer p with $1 \leq p \leq \min\{k, l\}$ such that $a_i \leq b_i \leq a_i$ for all $i = 1, 2, \ldots, p$, and either $p < \min\{k, l\}$ and $a_{p+1} < b_{p+1}$, or $p = k$ and $k \leq l$.

Lemma 2.2.1. *If $(Q, \leq)$ is a linear quasi-order, then $\leq$-domination is a linear quasi-order on the set of subsets of Q.*

Proof. It is obvious that $\leq$-domination is reflexive. Assume that B $\leq$-dominates A and C $\leq$-dominates B. Assume that the elements of A can be listed as $a_1 \geq a_2 \geq \cdots \geq a_k$, the elements of B can be listed as $b_1 \geq b_2 \geq \cdots \geq b_l$, and the elements of C can be listed as $c_1 \geq c_2 \geq \cdots \geq c_m$. By definition, there exists an integer p_1 with $1 \leq p_1 \leq \min\{k, l\}$ such that $a_i \leq b_i \leq a_i$ for all $i = 1, 2, \ldots, p_1$, and either $p_1 < \min\{k, l\}$ and $a_{p_1+1} < b_{p_1+1}$, or $p_1 = k \leq l$; and there exists an integer p_2 with $1 \leq p_2 \leq \min\{l, m\}$ such that $b_i \leq c_i \leq b_i$ for all $i = 1, 2, \ldots, p_2$, and either $p_2 < \min\{l, m\}$ and $b_{p_2+1} < c_{p_2+1}$, or $p_2 = l \leq m$.

Let $p = \min\{p_1, p_2\}$. Then $a_i \leq c_i \leq a_i$ for all $i = 1, 2, \ldots, p$. If either $p_1 < \min\{k, l\}$ and $a_{p_1+1} < b_{p_1+1}$, or $p_2 < \min\{l, m\}$ and $b_{p_2+1} < c_{p_2+1}$, then $p < \min\{k, m\}$ and $a_{p+1} < c_{p+1}$. If $p_1 = k \leq l$ and $p_2 = l \leq m$, then $p = k \leq m$. Therefore, C $\leq$-dominates A, and so $\leq$-domination is transitive.

Now let A, B be as above, and let p be the maximum integer such that $p \leq \min\{k, l\}$ and $a_i \leq b_i \leq a_i$ for all $i = 1, 2, \ldots, p$. Then if $p < \min\{k, l\}$, then A $\leq$-dominates B if $a_{p+1} > b_{p+1}$ and B $\leq$-dominates A if $a_{p+1} < b_{p+1}$. If $p = \min\{k, l\}$ then A $\leq$-dominates B if $k \geq l$ and B $\leq$-dominates A if $k \leq l$. Hence, $\leq$-domination is linear. $\square$

We say that B *strictly $\leq$-dominates* A if B $\leq$-dominates A in such a way that the numberings and integer p can be chosen in such a way that either $p < \min\{k, l\}$, or $p = k$ and $k < l$.

Lemma 2.2.2. *Let $(Q, \leq)$ be a linear quasi-order, let $A, B \subseteq Q$, and let B $\leq$-dominate A. Then B strictly $\leq$-dominates A if and only if A does not $\leq$-dominate B.*

Proof. Let p be as in the definition of B $\leq$-dominates A. Then $p < \min\{k, l\}$ and $a_{p+1} < b_{p+1}$, or $p = k \leq l$. Assume B strictly $\leq$-dominates A. If $p < \min\{k, l\}$ then $a_{p+1} < b_{p+1}$, so A does not $\leq$-dominate B. If $p = k < l$ then A also does not $\leq$-dominate B. Conversely, if A does not $\leq$-dominate B, then $p < \min\{k, l\}$ or $k < l$, so B strictly $\leq$-dominates A. $\square$

Let G be a graph and let P be a subgraph of G. By a P-bridge of G we mean a subgraph J of G such that either

- J is isomorphic to the complete graph on two vertices with $V(J) \subseteq V(P)$ and $E(J) \cap E(P) = \emptyset$, or
- J consists of a component of $G - V(P)$ together with all edges from that component to P.

We now define a linear quasi-order $\leq$ on the class of finite trees as follows. Let $n \geq 1$ be an integer, and suppose that $T \leq T'$ has been defined for all trees T on fewer than n vertices. Let T be a tree on n vertices, and let T' be an arbitrary tree. We define $T \leq T'$ if either $|V(T)| < |V(T')|$, or $|V(T)| = |V(T')|$ and for every maximal path P' of T' there exists a maximal path P of T such that the set of P'-bridges of T' $\leq$-dominates the set of P-bridges of T. It follows from Lemma 2.2.3 below that $\leq$ is indeed a linear quasi-order; in particular, it is well-defined.

If T, T' are trees, P is a path in T and P' is a path in T' we define $(T, P) \preceq (T', P')$ if either $|V(T)| < |V(T')|$, or $|V(T)| = |V(T')|$ and the set of P'-bridges of T' $\leq$-dominates the set of P-bridges of T.

Lemma 2.2.3. *(i) For every tree T there exists a maximal path $P(T)$ in T such that $(T, P(T)) \preceq (T, P)$ for every maximal path P in T.*
(ii) For every two trees T, T', we have $T \leq T'$ if and only if $(T, P(T)) \preceq (T', P(T'))$.
(iii) The ordering $\leq$ is a linear quasi-order on the class of finite trees.

Proof. We prove all three statements simultaneously by induction. Let $n \geq 1$ be an integer, assume inductively that all three statements have been proven for trees on fewer than n vertices, and let T be a tree on n vertices.
(i) Statement (i) clearly holds for one-vertex trees, and so we may assume that $n \geq 2$. Let $\mathcal{B}$ be the set of all P-bridges of T for all maximal paths P of T. Then every member of $\mathcal{B}$ has fewer than n vertices, and hence $\mathcal{B}$ is a linear quasi-order by $\leq$ by the induction hypothesis applied to (iii). By Lemma 2.2.1 the set of subsets of $\mathcal{B}$ is linearly quasi-ordered by $\leq$-domination. It follows that there exists a maximal path $P(T)$ in T such that the set of $P(T)$-bridges of T is minimal under $\leq$-domination.
(ii) The statement is obvious when $|V(T)| \neq |V(T')|$, so assume $n = |V(T)| = |V(T')|$, and let $\mathcal{B}$ be the set of all P-bridges of T for all maximal paths P of T and the set of all P'-bridges of T' for all maximal paths P' of T'. Then as in (i) the subsets of $\mathcal{B}$ are linearly quasi-ordered by $\leq$-domination. If $T \leq T'$, then by definition there exists a maximal path

P of T such that $(T, P) \preceq (T', P(T'))$. Hence $(T, P(T))) \preceq (T', P(T'))$ follows from (i). If $(T, P(T))) \preceq (T', P(T'))$, then by (i) $(T, P(T))) \preceq (T', P')$ for every maximal path P' in T', so $T \leq T'$.
(iii) Let T and T' be two trees. We may assume that $n = |V(T)| = |V(T')|$. Let $\mathcal{B}$ be as in (ii); then subsets of $\mathcal{B}$ are linearly quasi-ordered by $\leq$-domination. Then either $(T, P(T)) \preceq (T', P(T'))$ or $(T', P(T')) \preceq (T, P(T))$, and so by (ii) $\leq$ is linear. $\square$

For a tree T, the path $P(T)$ from Lemma 2.2.3(i) will be called a *spine of* T. For later application we need the following lemma.

Lemma 2.2.4. *Let T, T' be trees on the same number of vertices, let P' be a spine of T', and let P be a path in T. If the set of P'-bridges of T' strictly $\leq$-dominates the set of P-bridges of T, then $T < T'$.*

Proof. We have $(T, P) \preceq (T', P')$ and $(T', P') \not\preceq (T, P)$ by Lemma 2.2.2. Let P_1 be a maximal path that contains P; then $(T, P_1) \preceq (T, P)$. Therefore, $(T, P_1) \preceq (T', P')$ and $(T', P') \not\preceq (T, P_1)$. By Lemma 2.2.3(i), $(T, P(T)) \preceq (T, P_1) \preceq (T', P')$ and $(T', P') \not\preceq (T, P(T))$. By Lemma 2.2.3(ii), $T \leq T'$ and $T' \not\leq T$. Therefore, $T < T'$. $\square$

By a *rank* we mean a class of $\leq$-equivalent trees. If r is a rank we say that T *has rank* r or that the *rank of* T *is* r if $T \in r$. The class of all ranks will be denoted by $\mathcal{R}$.

Let T be a tree, and let t be a vertex of T. By a spine-decomposition of T relative to t we mean a sequence $(T_0, P_0, T_1, P_1, \ldots, T_l, P_l)$ such that

(i) $T_0 = T$,

(ii) for $i = 0, 1, \ldots, l$, P_i is a spine of T_i, and

(iii) for $i = 1, 2, \ldots, l$, $t \notin V(P_{i-1})$ and T_i is the P_{i-1}-bridge of T_{i-1} containing t.

Lemma 2.2.5. *Let T be a tree, let t be a vertex of T of degree three with neighbors t'_1, t'_2, t'_3, and let $(T_0, P_0, T_1, P_1, \ldots, T_l, P_l)$ be a spine-decomposition of T relative to t with $t \in$*

$V(P_l)$. Then exactly two of t_1', t_2', t_3' belong to $V(P_l)$, say t_1' and t_2'. Let r_3, r_3' be adjacent vertices of T such that r_3, r_3', t_3', t occur on a path of T in the order listed. Thus possibly $t_3' = r_3'$, but $t_3' \neq r_3$. Let T' be obtained from T by subdividing the edge r_3r_3' twice (let r_3'', r_3''' be the new vertices so that r_3', r_3'', r_3''', r_3 occur on a path of T' in the order listed), deleting the edge tt_1', contracting the edges tt_2' and tt_3' and adding an edge joining t_1' and r_3'''. Then T' has strictly smaller rank than T.

Proof. Let $T_0' = T'$ and for $i = 1, 2, \ldots, l$, let T_i' be the P_{i-1}-bridge of T_{i-1}' containing r_3'''. Let P' be the unique maximal path in T' with $V(P_l) - \{t, t_2'\} \cup \{r_3'\} \subseteq V(P')$. From the definition of a spine-decomposition and the fact that $t_3' \notin V(P_l)$ we deduce that $r_3 \in V(T_i)$ for all $i = 0, 1, \ldots, l$. It follows that $r_3 \in V(T_i')$ and $|V(T_i)| = |V(T_i')|$ for all $i = 0, 1, \ldots, l$. The P_l-bridge of T_l that contains r_3 is replaced by P'-bridges of T_l' with smaller cardinalities. Other P_l-bridges of T_l are unchanged in T'. Therefore, the set of P_l-bridges of T_l strictly $\leq$-dominates the set of P'-bridges of T_l', and hence $T_l' < T_l$ by Lemma 2.2.4. This implies, by induction on $l - i$ using Lemma 2.2.4, that $T_i' < T_i$ for all $i = 0, 1, \ldots, l$; that is, T' has smaller rank than T. □

2.3 A theorem about tree-decompositions

Let (T, Y) be a tree-decomposition of a graph G, let n be an integer, and let r be a rank. By an (n, r)–*cell* in (T, Y) we mean any component of the restriction of T to $\{t \in V(T) : |Y_t| \geq n\}$ that has rank at least r. Let us remark that if K is an (n, r)-cell in (T, Y) and $r \geq r'$, then K is an (n, r')-cell as well. The *size* of a tree-decomposition (T, Y) is the family of numbers

$$(1) \qquad (a_{n,r} : n \geq 0, r \in \mathcal{R}),$$

where $a_{n,r}$ is the number of (n, r)-cells in (T, Y). Sizes are ordered lexicographically; that is, if

$$(b_{n,r} : n \geq 0, r \in \mathcal{R}) \tag{2}$$

is the size of another tree-decomposition (R, Z) of the graph G, we say that (2) is *smaller than* (1) if there are an integer $n \geq 0$ and a rank $r \in \mathcal{R}$ such that $a_{n,r} > b_{n,r}$ and $a_{n',r'} = b_{n',r'}$ whenever either $n' > n$, or $n' = n$ and $r' > r$.

Lemma 2.3.1. *The relation "to be smaller than" is a well–ordering on the set of sizes of tree–decompositions of* G*.*

Proof. Since this ordering is clearly linear, it is enough to show that it is well–founded. Suppose for a contradiction that $\{(a^{(i)}_{n,r} : n \geq 0, r \in \mathcal{R})\}_{i=1}^{\infty}$ is a strictly decreasing sequence of sizes, and for $i = 1, 2, \ldots$, let n_i, r_i be such that $a^{(i)}_{n_i,r_i} > a^{(i+1)}_{n_i,r_i}$ and $a^{(i)}_{n,r} = a^{(i+1)}_{n,r}$ for (n, r) such that either $n > n_i$, or $n = n_i$ and $r > r_i$. Since $a^{(1)}_{n,r} = 0$ for all $r \in \mathcal{R}$ and all $n > |V(G)|$, we may assume (by taking a suitable subsequence) that $n_1 = n_2 = \cdots$, and that $r_1 \leq r_2 \leq r_3 \leq \cdots$. Since clearly $a^{(i)}_{n,r} \geq a^{(i)}_{n,r'}$ for all $n \geq 0$, all $r \leq r'$ and all $i = 1, 2, \ldots$, we have

$$a^{(1)}_{n_1,r_1} > a^{(2)}_{n_1,r_1} \geq a^{(2)}_{n_2,r_2} > a^{(3)}_{n_2,r_2} \geq a^{(3)}_{n_3,r_3} > \cdots ,$$

a contradiction. $\square$

We say that a tree-decomposition (T, W) of a graph G is *minimal* if there is no tree-decomposition of G of smaller size.

Lemma 2.3.2. *Let w be an integer, and let G be a graph of tree-width at most w. Then a minimal tree-decomposition of G exists, and every minimal tree-decomposition of G has width at most w.*

Proof. The existence of a minimal tree-decomposition follows from Lemma 2.3.1. If G has a tree-decomposition of width at most w, then every minimal tree-decomposition has width at most w, as desired. $\square$

Theorem 2.3.3. *Let (T, W) be a minimal tree-decomposition of a graph G. Then (T, W) satisfies* (W1)–(W6).

Proof. That (T, W) satisfies (W3) is shown in [28], and that it satisfies (W4), (W5) and (W6) is shown in [19]. Let us remark that [19] and [28] use a slightly different definition of minimality, but the proofs are adequate, because a minimal tree-decomposition in our sense is minimal in the sense of [19] and [28] as well. $\square$

Lemma 2.3.4. *Let (T, W) be a minimal tree-decomposition of a graph G. Then for every edge $tt' \in E(T)$ either $W_t \subseteq W_{t'}$ or $W_{t'} \subseteq W_t$.*

Proof. Assume for a contradiction that there exists an edge $tt' \in E(T)$ such that $W_t \not\subseteq W_{t'}$ and $W_{t'} \not\subseteq W_t$. Let R be obtained from T by subdividing the edge tt' and let t'' be the new vertex. Let $Y_{t''} = W_t \cap W_{t'}$ and $Y_r = W_r$ for all $r \in V(T)$, and let $Y = (Y_r : r \in V(R))$. Then (R, Y) is a tree-decomposition of G with smaller size than (T, W), contrary to the minimality of (T, W). $\square$

Lemma 2.3.5. *Let (T, W) be a minimal tree-decomposition of a graph G, let $t \in V(T)$, let $X \subseteq W_t$, let B be a component of $T \backslash t$, let t' be the neighbor of t in B, let $Y = \bigcup_{r \in V(B)} W_r$, and let H be the subgraph of G induced by $Y \cup W_t$. If $H \backslash X = H_1 \cup H_2$, where $V(H_1) \cap V(H_2) = \emptyset$ and both of $V(H_1), V(H_2)$ intersect W_t, then either $W_{t'} - X \subseteq W_t \cap V(H_1)$ or $W_{t'} - X \subseteq W_t \cap V(H_2)$.*

Proof. We first prove the following claim.

Claim 2.3.5.1. *Either $W_t \cap W_{t'} - X \subseteq V(H_1)$ or $W_t \cap W_{t'} - X \subseteq V(H_2)$.*

To prove the claim suppose for a contradiction that there exist vertices $v_1 \in W_t \cap W_{t'} \cap V(H_1)$ and $v_2 \in W_t \cap W_{t'} \cap V(H_2)$. Thus both v_1 and v_2 are B-tied, and so by (W6), which (T, W) satisfies by Theorem 2.3.3, there exists a B-confined path Q with ends v_1 and v_2. Since Q is B-confined, it is a subgraph of $H \backslash X$, contrary to the fact that $V(H_1) \cap V(H_2) = \emptyset$ and $H_1 \cup H_2 = H \backslash X$. This proves Claim 2.3.5.1.

Since both of $V(H_1), V(H_2)$ intersect W_t, Claim 2.3.5.1 implies that $W_t \not\subseteq W_{t'}$, and hence $W_{t'} \subseteq W_t$ by Lemma 2.3.4. By another application of Claim 2.3.5.1 we deduce that either $W_{t'} - X \subseteq W_t \cap V(H_1)$ or $W_{t'} - X \subseteq W_t \cap V(H_2)$, as desired. □

Lemma 2.3.6. *Let $k \geq 1$ be an integer, let (T, W) be a minimal tree-decomposition of a graph G, let $t_1, t_2 \in V(T)$, let $X = W_{t_1} \cap W_{t_2}$, let H be the subgraph of G induced by $\bigcup W_t$, the union taken over all vertices $t \in V(T)$ such that either $t \in \{t_1, t_2\}$, or for $i = 1, 2$ the vertex t belongs to the component of $T \backslash t_i$ containing t_{3-i}, let $H \backslash X = H_1 \cup H_2$, where $V(H_1) \cap V(H_2) = \emptyset$, and assume that $|W_{t_i} \cap V(H_j)| = k$ and $|W_t \cap V(H_i)| \geq k$ for all $i, j \in \{1, 2\}$ and all $t \in T[t_1, t_2]$. Let t, t' be two adjacent vertices on the path of T between t_1 and t_2. Then there exists an integer $i \in \{1, 2\}$ such that $W_t \cap V(H_i) = W_{t'} \cap V(H_i)$ and this set has cardinality k.*

Proof. We begin with the following claim.

Claim 2.3.6.1. *For every $t \in T[t_1, t_2]$ either $|W_t \cap V(H_1)| = k$ or $|W_t \cap V(H_2)| = k$.*

To prove the claim let R be the subtree of T induced by vertices $r \in V(T)$ such that either $r \in \{t_1, t_2\}$ or r belongs to the component of $T \backslash \{t_1, t_2\}$ that contains neighbors of both t_1 and t_2, let R_1, R_2 be two isomorphic copies of R, and for $r \in V(R)$ let r_1 and r_2 denote the copies of r in R_1 and R_2, respectively. Assume for a contradiction that there is $t_0 \in T[t_1, t_2]$ such that $|W_{t_0} \cap V(H_i)| > k$ for all $i \in \{1, 2\}$, and choose such a vertex with $t_0 \in V(R)$ and $|W_{t_0}|$ maximum. We construct a new tree-decomposition (T', W') as follows. The tree T' is obtained from the disjoint union of $T \backslash (V(R) - \{t_1, t_2\})$, R_1 and R_2 by identifying t_1 with $(t_1)_1$, $(t_2)_1$ with $(t_1)_2$ and $(t_2)_2$ with t_2 (here $(t_1)_2$ denotes the copy of t_1 in R_2 and similarly for the other three quantities). The family $W' = (W'_t : t \in V(T'))$ is defined as follows:

$$W'_t = \begin{cases} W_t & \text{if } t \in V(T) - V(R) \\ (W_r \cap V(H_1)) \cup (W_{t_1} \cap V(H_2) \cup X & \text{if } t = r_1 \text{ for } r \in T[t_1, t_2] \\ (W_r \cap V(H_2)) \cup (W_{t_2} \cap V(H_1) \cup X & \text{if } t = r_2 \text{ for } r \in T[t_1, t_2] \\ W_r \cap V(H_1) & \text{if } t = r_1 \text{ for } r \in V(R) - T[t_1, t_2] \\ W_r \cap V(H_2) & \text{if } t = r_2 \text{ for } r \in V(R) - T[t_1, t_2] \end{cases}$$

Please note that the value of W'_t is the same for $t = (t_2)_1$ and $t = (t_1)_2$, and hence W' is well-defined. Since no edge of G has one end in $V(H_1)$ and the other end in $V(H_2)$, it follows that (T', W') is a tree-decomposition of G.

We claim that the size of (T', W') is smaller than the size of (T, W). Indeed, let $n_0 = |W_{t_0}|$, and let $Z = \{t \in V(T') : |W'_t| \geq n_0\}$. Then $n_0 > 2k + |X|$. We define a mapping $f : Z \to V(T)$ by $f(t) = t$ for $t \in Z - V(R_1) - V(R_2)$, $f(r_1) = r$ for $r \in V(R)$ such that $r_1 \in Z$ and $f(r_2) = r$ for $r \in V(R)$ such that $r_2 \in Z$. We remark that the vertex obtained by identifying $(t_2)_1$ with $(t_1)_2$ does not belong to Z, and hence there is no ambiguity. Then Z and f have the following properties:

- $|W_{f(t)}| \geq |W'_t|$ for every $t \in Z$,
- for $r \in V(R)$, at most one of r_1, r_2 belongs to Z, and
- $(t_0)_1, (t_0)_2 \notin Z$

These properties follow from the assumptions that $|W_{t_i} \cap V(H_j)| = k$ and $|W_t \cap V(H_i)| \geq k$ for all $i, j \in \{1, 2\}$ and all $t \in T[t_1, t_2]$. (To see the second property assume for a contradiction that for some $r \in V(R)$ both r_1 and r_2 belong to Z. Then $n_0 = |W_{t_0}| \geq |W_{f(r_i)}| \geq |W_{r_i}| \geq n_0$, by the maximality of $|W_{t_0}|$ and the first property, and so equality holds throughout, contrary to the construction.) It follows from the first two properties that f maps injectively (n, r)-cells in (T', W') to (n, r)-cells in (T, W) for all $n \geq n_0$ and all ranks r. On the other hand, the third property implies that, letting r_1 denote the rank of one-vertex trees, no (n_0, r_1)-cell in (T', W') is mapped onto the (n_0, r_1)-cell in (T, W) with vertex-set $\{t_0\}$. Thus the size of (T', W') is smaller than the size of (T, W), contrary to the minimality of (T, W). This proves Claim 2.3.6.1.

Now let $t, t' \in T[t_1, t_2]$ be adjacent. By Lemma 2.3.4 we may assume that $W_t \subseteq W_{t'}$. Then $W_t \cap V(H_1) \subseteq W_{t'} \cap V(H_1)$ and $W_t \cap V(H_2) \subseteq W_{t'} \cap V(H_2)$. By Claim 2.3.6.1 we may assume that $|W_{t'} \cap V(H_1)| = k$. Given that $|W_t \cap V(H_1)| \geq k$ we have $W_t \cap V(H_1) = W_{t'} \cap V(H_1)$ and this set has cardinality k, as desired. $\square$

Lemma 2.3.7. *Let (T, W) be a minimal tree-decomposition of a graph G, let t_1, t_2, t_3 be a W-separable triad in T with center t_0, and let X, H, H_1, H_2 and H_3 be as in the definition of W-separable triad. Let $k = |W_{t_1} - X|/2$ and for $i = 1, 2, 3$ let t'_i denote the neighbor of t_0 in the component of $T \backslash t_0$ containing t_i. Then for all distinct $i, j \in \{1, 2, 3\}$, $V(H_i) \cap W_{t'_j} = V(H_i) \cap W_{t_0}$, and this set has cardinality k.*

Proof. Let $X_3 = \bigcup W_t$, the union taken over all $t \in V(T)$ that do not belong to the component of $T \backslash t_3$ containing t_0. Since $|W_{t_0} \cap V(H_1)| \geq k$ and $|W_{t_0} \cap V(H_2)| \geq k$ by the definition of W-separable triad, by Lemma 2.3.6 applied to t_1, t_2, H_3 and the subgraph of G induced by $V(H_1) \cup V(H_2) \cup X_3$ we deduce that $V(H_3) \cap W_{t_0} = V(H_3) \cap W_{t'_1} = V(H_3) \cap W_{t'_2}$, and this set has cardinality k. Similarly we deduce that $V(H_2) \cap W_{t_0} = V(H_2) \cap W_{t'_1} = V(H_2) \cap W_{t'_3}$ and $V(H_1) \cap W_{t_0} = V(H_1) \cap W_{t'_2} = V(H_1) \cap W_{t'_3}$, and that the latter two sets also have cardinality k. $\square$

We are finally ready to prove Theorem 2.1.4, which, by Lemma 2.3.2 is implied by the following theorem.

Theorem 2.3.8. *Let (T, W) be a minimal tree-decomposition of a graph G. Then (T, W) satisfies* (W1)–(W7).

Proof. That (T, W) satisfies (W1)–(W6) follows from Theorem 2.3.3. Thus it remains to show that (T, W) satisfies (W7). Suppose for a contradiction that (T, W) does not satisfy (W7), and let t_1, t_2, t_2 be a W-separable triad in T with center t_0 such that $W_{t_i} \cap W_{t_0} \subseteq X$ for every $i = 1, 2, 3$, where $X = W_{t_1} \cap W_{t_2} \cap W_{t_3}$. Let H, H_1, H_2 and H_3 be as in the definition of W-separable triad, and for $i \in \{1, 2, 3\}$ let t'_i denote the neighbor of t_0 in the component of $T \backslash t_0$ containing t_i.

Let $n := |W_{t_1}|$, let $k := |W_{t_1} - X|/2$, let r_1 denote the rank of 1-vertex trees, and let T_0 denote the (n, r_1)-cell containing t_0. By the definition of W-separable triad we have $|W_{t'_i}| \geq n$ for all $i \in \{1, 2, 3\}$, and hence the degree of t_0 in T_0 is at least three and by Lemmas 2.3.7 and 2.3.5 it is at most three.

Let $(T_0, P_0, T_1, P_1, \ldots, T_l, P_l)$ be a spine-decomposition of T_0 relative to t_0 with $t_0 \in V(P_l)$. Since P_l is a maximal path in T_l we may assume that $t'_1, t'_2 \in V(P_l)$ and $t'_3 \notin V(P_l)$.

It follows from Lemma 2.3.7 that $W_{t_3} \cap W_{t'_3} = X$. By Lemma 2.3.6 applied to t_3 and t'_3 and t'_3 and its neighbor in $T[t_3, t'_3]$ we deduce that there exists a vertex $r_3 \in T[t_3, t'_3] - \{t'_3\}$ such that either $V(H_1) \cap W_{t'_3} = V(H_1) \cap W_r$ for every $r \in T[r_3, t'_3]$, or $V(H_2) \cap W_{t'_3} = V(H_2) \cap W_r$ for every $r \in T[r_3, t'_3]$. Without loss of generality we may assume the latter. We may choose r_3 to be as close to t_3 as possible. The fact that $W_{t_3} \cap W_{t'_3} = X$ implies that $r_3 \neq t_3$. By another application of Lemma 2.3.6, this time to t_3, t'_3, r_3 and the neighbor of r_3 in $T[r_3, t_3]$, we deduce that $|V(H_1) \cap W_{r_3}| = |V(H_2) \cap W_{r_3}| = k$.

Let r'_3 be the neighbor of r_3 in $T[r_3, t_0]$ and let the tree T'' be defined as follows: for every component B of $T \backslash T[t_0, r'_3]$ not containing t_1, t_2 or t_3 let $r(B)r'(B)$ denote the edge connecting B to $T[t_0, r'_3]$, where $r(B) \in V(B)$ and $r'(B) \in T[t_0, r'_3]$. By Lemma 2.3.5 there exists an integer $i \in \{1, 2, 3\}$ such that $W_{r(B)} \subseteq W_{r'(B)} \cap V(H_i)$. Let us mention in passing that this, the choice of r_2 and Lemma 2.3.7 imply that for every such component B, every (n, r_1)-cell is either a subgraph of B or is disjoint from B. The tree T'' is obtained from T by, for every such component B for which either $i = 2$, or $i = 3$ and $r'(B) = t_0$, deleting the edge $r(B)r'(B)$ and adding the edge $t'_1 r(B)$; and for every such component

B for which $i = 1$ and $r'(B) = t_0$ deleting the edge $r(B)r'(B)$ and adding the edge $t_2'r(B)$. Since $W_{r'(B)} \cap (V(H_2) \cup V(H_3)) \subseteq W_{t_1'}$ by the choice of r_3 and Lemma 2.3.7; and $W_{r'(B)} \cap V(H_1) \subseteq W_{t_2'}$ by Lemma 2.3.7 it follows that (T'', W) is a tree-decomposition of G.

Let T' be defined as in Lemma 2.2.5, starting from the tree T'', let t_0' be the vertex that resulted from contracting the edges t_0t_2' and t_0t_3', and let $W' = (W_t' \mid t \in V(T'))$ be defined by

$$W_t' = \begin{cases} W_t & \text{if } t \in V(T') - T'[r_3''', t_0'] \\ W_{r_3} \cup (V(H_3) \cap W_{t_0}) & \text{if } t = r_3''' \\ (W_{r_3} - V(H_2)) \cup (V(H_3) \cap W_{t_0}) & \text{if } t = r_3'' \\ W_{t_2'} & \text{if } t = t_0' \\ (W_t - V(H_2)) \cup (V(H_3) \cap W_{t_0}) & \text{if } t \in T'[r_3', t_0'] - \{t_0'\} \end{cases}$$

We claim that (T', W') is a tree decomposition of G. Indeed, since $V(H_2) \cap W_r \subseteq W_{t_0}$ for all $r \in T[r_3', t_0]$ it follows that (T', W') satisfies (W1).

To show that (T', W') satisfies (W2) let $v \in V(G)$, let $Z = \{t \in V(T) : v \in W_t\}$. and let $Z' = \{t \in V(T') : v \in W_t'\}$. It suffices to show that Z' induces a connected subset of T', for this is easily seen to be equivalent to (W2). To that end assume first that $v \notin W_{t_1'} = W_{t_1'}' = W_{t_0} \cap (V(H_2) \cup V(H_3))$. It follows that, since Z induces a subtree of T, that Z' induces a subtree of T'. We assume next that $v \in W_{t_0} \cap V(H_2)$. The choice of T'' and the definition of W' imply that no vertex in the component of $T' \backslash r_3'''$ containing t_0' belongs to Z'. Again, it follows that Z' induces a subtree of T'. Finally, let $v \in W_{t_0} \cap V(H_3)$. Then $T'[t_1', t_0'] \subseteq Z'$, and it again follows that Z' induces a subtree of T'. This proves our claim that (T', W') is a tree-decomposition.

We claim that the size of (T', W') is smaller than the size of (T, W). Let r denote the rank of T_0, and let T_0' denote the (n, r_1)-cell in (T', W') containing t_0'. First, by the passing remark made a few paragraphs ago, for every integer $m \geq n$ and every rank s, to every (m, s)-cell in (T', W') other than T'_0 there corresponds a unique (m, s)-cell in (T, W). (To the $(n+1, r_1)$-cell in (T', W') with vertex-set $\{r'''_3\}$ there corresponds the $(n+1, r_1)$-cell in (T, W) with vertex-set $\{t_0\}$.) Second, by Lemma 2.2.5 the rank of T_0 is strictly larger than the rank of T'_0. Thus no (n, r)-cell in (T', W') corresponds to T_0. It follows that (T', W') is a tree-decomposition of G of smaller size, contrary to the minimality of (T, W). $\square$

CHAPTER 3

MINORS OF 2-CONNECTED GRAPHS OF LARGE PATH-WIDTH

The chapter is organized as follows. In Section 3.1 we prove that Theorem 1.1.4 follows from the main result of this chapter, Theorem 1.1.3. In Section 3.2 we introduce "cascades", our main tool, and prove that in any tree-decomposition with no duplicate bags of bounded width of a graph of big path-width there is an "injective" cascade of large height. In Section 3.3 we prove that every 2-connected graph of big path-width and bounded tree-width admits a tree-decomposition of bounded width and a cascade with linkages that are minimal. In Section 3.4 we analyze those minimal linkages and prove that there are essentially only two types of linkage. This is where we use the properties of tree-decompositions from Chapter 2. Finally, in Section 3.5 we convert the two types of linkage into the two families of graphs from Theorem 1.1.3.

3.1 Proof of Theorem 1.1.4

In this section we prove that Theorem 1.1.4 is implied by Theorem 1.1.3. Recall that in the introduction we define A as the graph that consists of a cycle $a_1a_2a_3a_4a_5a_6a_1$ and extra edges a_1a_3, a_3a_5, a_5a_1 and $C_{3,2}$ as the graph consisting of two disjoint triangles.

Lemma 3.1.1. *If a graph H has no $K_4, C_{3,2}$, or A minor, then H has a vertex v such that $H\backslash v$ is a forest.*

Proof. We proceed by induction on $|VH)|$. The lemma clearly holds when $|V(H)| = 0$, and so we may assume that H has at least one vertex and that the lemma holds for graphs on fewer than $|V(H)|$ vertices. If H has a vertex of degree at most one, then the lemma follows by induction by deleting such vertex. We may therefore assume that H has minimum degree at least two.

If H has a cutvertex, say v, then v is as desired, for if C is a cycle in $H\backslash v$, then $H\backslash V(C)$ also contains a cycle (because H has minimum degree at least two), and hence H has a $C_{3,2}$ minor, a contradiction. We may therefore assume that H is 2-connected.

We may assume that H is not a cycle, and hence it has an ear-decomposition $H = H_0 \cup H_1 \cup \cdots \cup H_k$, where $k \geq 1$, H_0 is a cycle and for $i = 1, 2, \ldots, k$ the graph H_i is a path with ends $u_i, v_i \in V(H_0 \cup H_1 \cup \cdots \cup H_{i-1})$ and otherwise disjoint from $H_0 \cup H_1 \cup \cdots \cup H_{i-1}$. If $u_1 \in \{u_i, v_i\}$ for all $i \in \{2, 3, \ldots, k\}$, then u_1 satisfies the conclusion of the lemma, and similarly for v_1. We may therefore assume that there exist $i, j \in \{2, 3, \ldots, k\}$ such that $u_1 \notin \{u_i, v_i\}$ and $v_1 \notin \{u_j, v_j\}$. It follows that H has a $K_4, C_{3,2}$, or A minor, a contradiction. $\square$

Lemma 3.1.2. *If a graph H has a vertex v such that $H\backslash v$ is a forest, then there exists an integer k such that H is isomorphic to a minor of $\mathcal{P}_k$.*

Proof. Let v be such that $T := H\backslash v$ is a forest. We may assume, by replacing H by a graph with an H minor, that T is isomorphic to CT_t for some t, and that v is adjacent to every vertex of T. It follows that H is isomorphic to a minor of $\mathcal{P}_{2t}$, as desired. $\square$

Lemma 3.1.3. *Let H be a 2-connected outerplanar near-triangulation with k triangles. Then H is isomorphic to a minor of $\mathcal{Q}_k$. Furthermore, the minor inclusion can be chosen in such a way that for every edge $a_1a_2 \in E(H)$ incident with the unbounded face and for every $i \in \{1, 2\}$, the vertex w_i belongs to the node of a_i, where w_1w_2 is the base edge of $\mathcal{Q}_k$.*

Proof. We proceed by induction on k. The lemma clearly holds when $k = 1$, and so we may assume that H has at least two triangles and that the lemma holds for graphs with fewer than k triangles. The edge a_1a_2 belongs to a unique triangle, say a_1a_2c. The triangle a_1a_2c divides H into two near-triangulations H_1 and H_2, where the edge a_ic is incident with the unbounded face of H_i. Let $Q_1, Q_2, u_1, v_1, u_2, v_2, w_1, w_2$ be as in the definition of $\mathcal{Q}_k$. By the induction hypothesis the graph H_i is isomorphic to a minor of Q_i in such a way that the vertex u_i belongs to the node of a_i and the vertex v_i belongs to the node of c. It follows that H is isomorphic to $\mathcal{Q}_k$ in such a way that w_i belongs to the node of a_i. $\square$

Lemma 3.1.4. *Let H be a graph that has no K_4 or $K_{2,3}$ minor. Then there exists an integer k such that H is isomorphic to a minor of $\mathcal{Q}_k$.*

Proof. It is well-known [13, Exercise 23] that the hypotheses of the lemma imply that H is outerplanar. We may assume, by replacing H by a graph with an H minor, that H is a 2-connected outerplanar near-triangulation. The lemma now follows from Lemma 3.1.3. $\square$

Corollary 3.1.5. *Let H be a graph that has no K_4, $K_{2,3}$, $C_{3,2}$, or A minor. Then there exists an integer k such that H is isomorphic to a minor of $\mathcal{P}_k$ and H is isomorphic to a minor of $\mathcal{Q}_k$.*

Proof. This follows from Lemmas 3.1.1, 3.1.2 and 3.1.4. $\square$

Proof of Theorem 1.1.4, assuming Theorem 1.1.3. To prove the"if" part notice that $\mathcal{P}_k$ and $\mathcal{Q}_k$ are 2-connected and have large path-width when k is large, because $\mathcal{Q}_k$ has a CT_{k-1} minor. There is no vertex v in A such that $A\backslash v$ is acyclic. So, A and $C_{3,2}$ are not minors of $\mathcal{P}_k$ for any k. The graph $\mathcal{Q}_k$ is outerplanar, so K_4 and $K_{2,3}$ are not minors of $\mathcal{Q}_k$ for any positive integer k. This means $g(H) \geq 3$ for $H \in \{K_4, K_{2,3}, C_{3,2}, A\}$. This proves the "if" part.

To prove the "only if" part, if H has no $K_4, K_{2,3}$, $C_{3,2}$ or A minor, then by Corollary 3.1.5 H is a minor of both $\mathcal{P}_k$ and $\mathcal{Q}_k$ for some k. Then $g(H) \leq 2$ by Theorem 1.1.3. $\square$

3.2 Cascades

In this section we introduce "cascades", our main tool. The main result of this section, Lemma 3.2.6, states that in any tree-decomposition with no duplicate bags of bounded width of a graph of big path-width there is an "injective" cascade of large height.

Lemma 3.2.1. *Let p, w be two positive integers and let G be a graph of tree-width strictly less than w and path-width at least p. Then for every tree-decomposition (T, X) of G of width strictly less than w, the path-width of T is at least $\lfloor p/w \rfloor$.*

Proof. We will prove the contrapositive. Assume there exists a tree-decomposition (T, X) of G of width $< w$ such that the path-width of T is less than $\lfloor p/w \rfloor$. Because the path-width of T is less than $\lfloor p/w \rfloor$, there exists a path-decomposition $(Y_1, Y_2, ..., Y_s)$ of T with $|Y_i| \leq \lfloor p/w \rfloor$ for all i. We will construct a path-decomposition $(Z_1, Z_2, ..., Z_s)$ for G with width less than p. Set $Z_i = \bigcup_{y \in Y_i} X_y$ for every $i \in \{1, 2, ..., s\}$. For every vertex $v \in V(G)$, v belongs to at least one set X_t for some $t \in V(T)$. The vertex t of tree T must be in Y_l for some $l \in \{1, 2, ..., s\}$, so $v \in X_t \subseteq Z_l$. Therefore, $\bigcup Z_i = V(G)$. Similarly, for every edge $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in X_t$. Therefore, $u, v \in Z_l$ for some $l \in \{1, 2, ..., s\}$.

Now, if a vertex $v \in V(G)$ belongs to both Z_a and Z_b for some $a, b \in \{1, 2, ..., s\}, a < b$, we will show that $v \in Z_c$ for all c such that $a < c < b$. Let c be an arbitrary integer satisfying $a < c < b$. The fact that $v \in Z_a$ implies $v \in X_{y_1}$ for some $y_1 \in Y_a$. Similarly, $v \in X_{y_2}$ for some $y_2 \in Y_b$. Let H be the set of vertices of T on the path from y_1 to y_2. Since $y_1 \in Y_a$ and $y_2 \in Y_b$, $H \cap Y_a \neq \emptyset \neq H \cap Y_b$. Hence, by Lemma 2.1.1 with $H = T$ and (T, Y) the path-decomposition $(Y_1, Y_2, ..., Y_s)$, we have $H \cap Y_c \neq \emptyset$. Let $t \in H \cap Y_c$, then $v \in X_t \subseteq Z_c$. So $(Z_1, Z_2, ..., Z_s)$ is a path-decomposition of G. Since the width of (T, X) is less than w, we have $|X_y| \leq w$ for every $y \in Y_i$, where $i \in \{1, 2, ..., s\}$. Therefore, $|Z_i| \leq w.\lfloor p/w \rfloor \leq p$ for every $i \in \{1, 2, ..., s\}$. Therefore, the width of $(Z_1, Z_2, ..., Z_s)$ is less than p, so the path-width of G is less than p, as desired. $\square$

Since CT_a has maximum degree at most three, the following lemma follows from [17, Lemma 6].

Lemma 3.2.2. *Let T be a forest with path-width at least $a \geq 1$. Then there exists a homeomorphic embedding $CT_{a-1} \hookrightarrow T$.*

Let $\eta : T \hookrightarrow T'$. We define $sp(\eta)$, the *span of* η, to be the set of vertices $t \in V(T')$ that lie on the path from $\eta(t_1)$ to $\eta(t_2)$ for some vertices $t_1, t_2 \in V(T)$.

Let $s > 0$ be an integer and let (T, X) be a tree-decomposition of a graph G. By a *cascade of height* h *and size* s *in* (T, X) we mean a homeomorphic embedding $\eta : T_h \hookrightarrow T$ such that $|X_{\eta(t)}| = s$ for every minor vertex $t \in V(T_h)$ and $|X_t| \geq s$ for every t in the span of η.

Lemma 3.2.3. *For any positive integer h and nonnegative integers a, k, the following holds. Let $m = (a+2)h + a$. Let (T, X) be a tree-decomposition of a graph G and let $\phi : CT_m \hookrightarrow T$ be a homeomorphic embedding such that $|X_t| \geq k$ for all $t \in sp(\phi)$. If for every $t \in V(CT_m)$ at height $l \leq m - a$ there exist a descendant t' of t at height $l + a$ and a vertex $r \in T[\phi(t), \phi(t')]$ such that $|X_r| = k$, then there exists a cascade η of height h and size k in (T, X).*

Proof. By hypothesis there exist a vertex $x_0 \in V(CT_m)$ at height a and a vertex $u_0 \in V(T)$ on the path from the image under ϕ of the root of CT_m to $\phi(x_0)$ such that $|X_{u_0}| = k$. Let x be a child of x_0, and let x_1 and x_2 be the children of x. By hypothesis there exist, for $i = 1, 2$, a vertex $y_i \in V(CT_m)$ at height $2a + 2$ that is a descendant of x_i and a vertex $u_i \in T[\phi(x_i), \phi(y_i)]$ such that $|X_{u_i}| = k$. Let r be the major root of T_1, and let (t_0, t_1, t_2) be its trinity. We define $\eta_1 : T_1 \hookrightarrow T$ by $\eta_1(t_i) = u_i$ for $i = 0, 1, 2$ and $\eta_1(r) = \phi(x)$. Then η_1 is a cascade of height one and size k in (T, X). If $h = 1$, then η_1 is as desired, and so we may assume that $h > 1$.

Assume now that for some positive integer $l < h$ we have constructed a cascade $\eta_l : T_l \hookrightarrow T$ of height l and size k in (T, X) such that for every leaf t_0 of T_l other than the minor root there exists a vertex $x_0 \in V(CT_m)$ at height $(a+2)l + a$ such that the image under η_l of every vertex on the path in T_l from the minor root to t_0 belongs to the path in T from the image under ϕ of the root of CT_m to $\phi(x_0)$. Our objective is to extend η_l to a cascade η_{l+1} of height $l + 1$ and size k in (T, X) with the same property. To that end let $\eta_{l+1}(t) = \eta_l(t)$ for all $t \in V(T_l)$, let t_0 be a leaf of T_l other than the minor root and let x_0

be as earlier in the paragraph. Let x be a child of x_0, and let x_1 and x_2 be the children of x. By hypothesis there exist, for $i = 1, 2$, a vertex $y_i \in V(CT_m)$ at height $(a+2)(l+1)+a$ that is a descendant of x_i and a vertex $u_i \in T[\phi(x_i), \phi(y_i)]$ such that $|X_{u_i}| = k$. Let r be the child of t_0 in T_{l+1}, and let (t_0, t_1, t_2) be its trinity. We define $\eta_{l+1}(t_i) = u_i$ for $i = 1, 2$ and $\eta_{l+1}(r) = \phi(x)$. This completes the definition of η_{l+1}.

Now η_h is as desired. □

Lemma 3.2.4. *For any two positive integers h and w, there exists a positive integer $p = p(h, w)$ such that if G is a graph of path-width at least p, then in any tree-decomposition of G of width less than w, there exists a cascade of height h.*

Proof. Let $a_{w+1} = 0$, and for $k = w, w-1, \ldots, 0$ let $a_k = (a_{k+1}+2)h + a_{k+1}$, and let $p = w(a_0 + 1)$. We claim that p satisfies the conclusion of the lemma. To see that let (T, X) be a tree-decomposition of G of width less than w. Let $k \in \{0, 1, \ldots, w+1\}$ be the maximum integer such that there exists a homeomorphic embedding $\phi : CT_{a_k} \hookrightarrow T$ satisfying $|X_t| \geq k$ for all $t \in sp(\phi)$. Such an integer exists, because $k = 0$ satisfies those requirements by Lemmas 3.2.1 and 3.2.2, and it satisfies $k \leq w$, because the width of (T, X) is less than w. The maximality of k implies that for the integers h, k and a_{k+1} the hypothesis of Lemma 3.2.3 is satisfied. Thus the lemma follows from Lemma 3.2.3. □

Let (T, X) be a tree-decomposition of a graph G, and let $\eta : T_h \hookrightarrow T$ be a cascade of height h and size s in (T, X). We say that η is *injective* if there exists $I \subseteq V(G)$ such that $|I| < s$ and $X_{\eta(t)} \cap X_{\eta(t')} = I$ for every two distinct vertices $t, t' \in V(T_h)$. We call this set I the *common intersection set* of η.

Lemma 3.2.5. *Let a, b, s, w be positive integers and let k be a nonnegative integer. Let (T, X) be a tree-decomposition of a graph G of width strictly less than w. Let $h =(2(a+2)w+2)b$. If there is a cascade η of height h and size $s+k$ in (T, X) such that $|\bigcap_{t \in V(T_h)} X_{\eta(t)}| \geq k$, then either there is a cascade η' of height a and size $s+k$ in (T, X) such that $|\bigcap_{t \in V(T_a)} X_{\eta'(t)}| \geq$*

$k+1$ *or there is an injective cascade* η' *of height* b*, size* $s+k$ *and common intersection set of size* k *in* (T, X).

Proof. We may assume that

($*$) there does not exist a cascade η' of height a and size $s+k$ in (T, X) such that $|\bigcap_{t \in V(T_a)} X_{\eta'(t)}| \geq k+1$.

Let $F = \bigcap_{t \in V(T_h)} X_{\eta(t)}$. By ($*$), $|F| = k$. We claim the following.

Claim 3.2.5.1. *For every vertex* $t \in T_h$ *at height* $l \leq h-a-2$ *and every* $u \in X_{\eta(t)} - F$ *there exists a descendant* $t' \in V(T_h)$ *of* t *at height at most* $l+a+2$ *such that* $u \notin X_{\eta(t')}$.

To prove the claim let $u \in X_{\eta(t)} - F$. By ($*$) in the subtree of T_h consisting of t and its descendants there is a vertex t' of height at most $l+a+2$ such that $u \notin X_{\eta(t')}$. This proves the claim.

We use the previous claim to deduce the following generalization.

Claim 3.2.5.2. *For every vertex* $t \in V(T_h)$ *at height* $l \leq h-(a+2)w$ *there exists a descendant* $t' \in V(T)$ *of* t *at height at most* $l+(a+2)w$ *such that* $X_{\eta(t)} \cap X_{\eta(t')} = F$.

To prove the claim let $X_{\eta(t)} \backslash F = \{u_1, u_2, \ldots, u_p\}$, where $p \leq w$. By Claim 3.2.5.1 there exists a descendant $t_1 \in V(T)$ of t at height at most $l+a+2$ such that $u_1 \notin X_{\eta(t')}$. By another application of Claim 3.2.5.1 there exists a descendant $t_2 \in V(T)$ of t_1 at height at most $l+2(a+2)$ such that $u_2 \notin X_{\eta(t')}$. By (W2) $u_1 \notin X_{\eta(t')}$. By continuing to argue in the same way we finally arrive at a vertex t_p that is a descendant of t at height at most $l+(a+2)p$ such that $X_{\eta(t)} \cap X_{\eta(t_p)} = F$. Thus t_p is as desired. This proves the claim.

Let $x_0 \in V(T_h)$ be the minor root of T_h. By Claim 3.2.5.2 and (W2) there exists a major vertex $x \in V(T)$ at height at most $(a+2)w+1$ such that $X_{\eta(x_0)} \cap X_{\eta(x)} = F$. Let y_1 and y_2 be the children of x. By Claim 3.2.5.2 and (W2) there exists, for $i = 1, 2$, a minor vertex $x_i \in V(T_h)$ at height at most $2(a+2)w+2$ that is a descendant of y_i and such that

$X_{\eta(x_i)} \cap X_{\eta(x)} = F$. Let r be the major root of T_1, and let (t_0, t_1, t_2) be its trinity. We define $\eta_1 : T_1 \hookrightarrow T$ by $\eta_1(t_i) = \eta(x_i)$ for $i = 0, 1, 2$ and $\eta_1(r) = \eta(x)$. Then η_1 is an injective cascade of height one and size $s + k$ in (T, X) with common intersection set F. If $b = 1$, then η_1 is as desired, and so we may assume that $b > 1$.

Assume now that for some positive integer $l < b$ we have constructed an injective cascade $\eta_l : T_l \hookrightarrow T$ of height l and size $s + k$ with common intersection set F in (T, X) such that for every leaf t_0 of T_l other than the minor root there exists a vertex $x_0 \in V(T_h)$ at height $(2(a + 2)w + 2)l$ such that the image under η_l of every vertex on the path in T_l from the minor root to t_0 belongs to the path in T from the image under η of the root of T_h to $\eta(x_0)$. Our objective is to extend η_l to an injective cascade η_{l+1} of height $l + 1$, size $s + k$, and common intersection set F in (T, X) with the same property. To that end let $\eta_{l+1}(t) = \eta_l(t)$ for all $t \in V(T_l)$, let t_0 be a leaf of T_l other than the minor root, and let x_0 be as earlier in the paragraph. By Claim 3.2.5.2 and (W2) there exists a descendant x of x_0 at height at most $(2(a+2)w+2)l+(a+2)w+1$ such that x is major and $X_{\eta_l(t_0)} \cap X_{\eta(x)} = F$. Let y_1 and y_2 be the children of x. By Claim 3.2.5.2 and (W2) there exists, for $i = 1, 2$, a minor vertex $x_i \in V(T_h)$ at height at most $(2(a + 2)w + 2)(l + 1)$ that is a descendant of y_i and such that $X_{\eta(x_i)} \cap X_{\eta(x)} = F$. Let r be the child of t_0 in T_{l+1}, and let (t_0, t_1, t_2) be its trinity. We define $\eta_{l+1}(t_i) = \eta(x_i)$ for $i = 1, 2$ and $\eta_{l+1}(r) = \eta(x)$. This completes the definition of η_{l+1}.

Now η_b is as desired. □

Lemma 3.2.6. *For any two positive integers h and w, there exists a positive integer $p = p(h, w)$ such that if G is a graph with tree-width less than w and path-width at least p, then in any tree-decomposition (T, X) of G that has width less than w and satisfies (W4), there is an injective cascade of height h.*

Proof. Let $a_w = 0$, and for $k = w - 1, \ldots, 0$ let $a_k = (2(a_{k+1} + 2)w + 2)h$. Let p be the integer in Lemma 3.2.4 for input integers a_0 and w. We claim that p satisfies the conclusion of the lemma. To see that let (T, X) be a tree-decomposition of G of width less than w satisfying (W4). By Lemma 3.2.4, there exists a cascade η of height a_0 in (T, X). Let $k \in \{0, 1, \ldots, w\}$ be the maximum integer such that there exists a cascade $\eta' : T_{a_k} \hookrightarrow T$ satisfying $|\bigcap_{t \in V(T_{a_k})} X_{\eta'(t)}| \geq k$. Such an integer exists, because $k = 0$ satisfies those requirements and $k < w$ because of (W4) and because the width of (T, X) is less than w. The maximality of k implies that there does not exist a cascade $\eta'' : T_{a_{k+1}} \hookrightarrow T$ satisfying $|\bigcap_{t \in V(T_{a_{k+1}})} X_{\eta''(t)}| \geq k + 1$. Thus the lemma follows from Lemma 3.2.5. $\square$

3.3 Ordered Cascades

The main result of this section, Theorem 3.3.5, states that every 2-connected graph of big path-width and bounded tree-width admits a tree-decomposition of bounded width and a cascade with linkages that are minimal.

Let (T, X) be a tree-decomposition of a graph G, and let η be an injective cascade in (T, X) with common intersection set I. Assume the size of η is $|I| + s$. Then we say η is *ordered* if for every minor vertex $t \in V(T_h)$ there exists a bijection $\xi_t : \{1, 2, \ldots, s\} \to X_{\eta(t)} - I$ such that for every major vertex t_0 with trinity (t_1, t_2, t_3), there exist s disjoint paths $P_1, P_2, \ldots, P_s$ in $G - I$ such that the path P_i has ends $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$, and there exist s disjoint paths $Q_1, Q_2, \ldots, Q_s$ in $G - I$ such that the path Q_i has ends $\xi_{t_1}(i)$ and $\xi_{t_3}(i)$. In that case we say that η is an *ordered cascade with orderings* ξ_t. We say that the set of paths $P_1, P_2, \ldots, P_s$ is a *left* t_0*-linkage with respect to* η, and that the set of paths $Q_1, Q_2, \ldots, Q_s$ is a *right* t_0*-linkage with respect to* η.

We will need to fix a left and a right t_0-linkage for every major vertex $t_0 \in V(T_h)$; when we do so we will indicate that by saying that η is an *ordered cascade in* (T, X) *with orderings* ξ_t *and specified linkages*, and we will refer to the specified linkages as the *left specified* t_0*-linkage* and the *right specified* t_0*-linkage*. We will denote the left specified t_0-linkage by $P_1(t_0), P_2(t_0), \ldots, P_s(t_0)$ and the right specified t_0-linkage by $Q_1(t_0), Q_2(t_0), \ldots, Q_s(t_0)$. We say that the specified t_0-linkages are *minimal* if for every set of disjoint paths P_1, P_2, $\ldots, P_s$ in $G - I$ from $X_{\eta(t_1)} - I$ to $X_{\eta(t_2)} - I$ such that $\xi_{t_1}(i)$ is an end of P_i (let the other end be p_i) and every set of disjoint paths $Q_1, Q_2, \ldots, Q_s$ in $G - I$ from $X_{\eta(t_1)} - I$ to $X_{\eta(t_3)} - I$ such that $\xi_{t_1}(i)$ is an end of Q_i (let the other end be q_i) we have

$$\left|E\left(\bigcup(x_iP_ip_i \cup x_iQ_iq_i)\right)\right| \geq \left|E\left(\bigcup(y_iP_i(t_0)\xi_{t_2}(i) \cup y_iQ_i(t_0)\xi_{t_3}(i))\right)\right|, \tag{3.1}$$

where the unions are taken over $i \in \{1, 2, ..., s\}$, x_i is the first vertex from $\xi_{t_1}(i)$ that P_i departs from Q_i, and y_i is the first vertex from $\xi_{t_1}(i)$ that $P_i(t_0)$ departs from $Q_i(t_0)$.

Lemma 3.3.1. *Let h and s be two positive integers, and let $\eta : T_h \hookrightarrow T$ be an injective cascade of height h and size s in a linked tree-decomposition (T, X) of a graph G. Then the cascade η can be turned into an ordered cascade with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_h)$.*

Proof. Let $s' := s - |I|$. To show that η can be made ordered let r be the minor root of T_h, let $\xi_r : \{1, 2, \ldots, s'\} \to X_{\eta(r)} - I$ be arbitrary, assume that for some integer $l \in \{0, 1, \ldots, h-1\}$ we have already constructed $\xi_t : \{1, 2, \ldots, s'\} \to X_{\eta(t)} - I$ for all minor vertices $t \in V(T_h)$ at height at most l, let $t \in V(T_h)$ be a minor vertex at height exactly l, let t_0 be its child, and let (t, t_1, t_2) be the trinity at t_0. By condition (W3) there exist s' disjoint paths $P_1, P_2, \ldots, P_{s'}$ in $G - I$ from $X_{\eta(t)} - I$ to $X_{\eta(t_1)} - I$ and s' disjoint paths $Q_1, Q_2, \ldots, Q_{s'}$ in $G - I$ from $X_{\eta(t)} - I$ to $X_{\eta(t_2)} - I$. We may assume that $\xi_t(i)$ is an end of P_i and Q_i and we define $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$ to be their other ends, respectively. We may also assume that these paths satisfy the minimality condition (3.1). It follows that η is an ordered cascade with orderings ξ_t and specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_h)$. $\square$

Lemma 3.3.2. *For every two integers $a \geq 1$ and $k \geq 1$ there exists an integer $h = h(a, k)$ such that the following holds. Color the major vertices of T_h using k colors. Then there exists a monotone homeomorphic embedding $\eta : T_a \hookrightarrow T_h$ such that the major vertices of T_a map to major vertices of the same color in T_h.*

Proof. Let c be one of the colors. We will prove by induction on k and subject to that by induction on b that there is a function $h = g(a, b, k)$ such that there is either a monotone homeomorphic embedding $\eta : T_a \hookrightarrow T_h$ such that the major vertices of T_a map to major vertices of the same color in T_h, or a monotone homeomorphic embedding $\eta : T_b \hookrightarrow T_h$ such that the major vertices of T_b map to major vertices of color c in T_h. In fact, we will show that $g(a, b, 1) = a$, $g(a, 1, k+1) \leq g(a, a, k)$ and $g(a, b+1, k+1) \leq g(a, b, k+1) + g(a, a, k)$.

The assertion holds for $k = 1$ by letting $h = a$ and letting η be the identity mapping. Assume the statement is true for some $k \geq 1$, let the major vertices of T_h be colored using $k+1$ colors, and let c be one of the colors. If $b = 1$, then if T_h has a major vertex colored c, then the second alternative holds; otherwise at most k colors are used and the assertion follows by induction on k.

We may therefore assume that the assertion holds for some integer $b \geq 1$ and we must prove it for $b + 1$. To that end we may assume that T_h has a major vertex t_0 colored c at height at most $g(a, a, k)$, for otherwise the assertion follows by induction on k. Let the trinity at t_0 be (t_1, t_2, t_3). For $i = 2, 3$ let R_i be the subtree of T_h with minor root t_i. If for some $i \in \{2, 3\}$ there exists a monotone homeomorphic embedding $T_a \hookrightarrow R_i$ such that the major vertices of T_a map to major vertices of the same color in T_h, then the statement holds. We may therefore assume that for $i \in \{2, 3\}$ there exists a monotone homeomorphic embedding $\eta_i : T_b^i \hookrightarrow R_i$ such that the major vertices of T_b^i map to major vertices of color c, the major root of T_{b+1} is r_0, the trinity at r_0 is (r_1, r_2, r_3) and T_b^i is the subtree of $T_{b+1} - \{r_0, r_1\}$ with minor root r_i. Let $\eta : T_{b+1} \hookrightarrow T_h$ be defined by $\eta(t) = \eta_i(t)$ for $t \in V(T_b^i)$, $\eta(r_0) = t_0$ and $\eta(r_1)$ is defined to be the minor root of T_h. Then $\eta : T_{b+1} \hookrightarrow T_h$ is as desired. This proves the existence of the function $g(a, b, k)$.

Now $h(a, k) = g(a, a, k)$ is as desired. □

Let (T, X) be a tree-decomposition of a graph G, and let $\eta : T_h \hookrightarrow T$ be an injective cascade in (T, X) with common intersection set I. Let $t_0 \in V(T_h)$ be a major vertex, and let (t_1, t_2, t_3) be the trinity at t_0. We define the *η-torso at t_0* as the subgraph of G induced by $\bigcup X_t - I$, where the union is taken over all t in $V(T)$ such that the unique path in T from t to $\eta(t_0)$ does not contain $\eta(t_1)$,$\eta(t_2)$, or $\eta(t_3)$ as an internal vertex.

Let $s > 0$ be an integer. Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) with size $|I|+s$ and orderings ξ_t, where I is the common intersection set of η. Let $t_0 \in V(T_h)$ be a major vertex, let (t_1, t_2, t_3) be the trinity at t_0, let G' be the η-torso at t_0, and let $i, j \in \{1, 2, \ldots, s\}$ be distinct. We say that t_0 *has property A_{ij} in η* if there exist disjoint tripods L_i, L_j in G' such that for each $m \in \{i, j\}$ the tripod L_m has feet $\xi_{t_1}(m), \xi_{t_2}(m_2), \xi_{t_3}(m_3)$ for some $m_2, m_3 \in \{i, j\}$.

We say that t_0 *has property B_{ij} in η* if there exist vertices $v_{x,y}$ for all $x \in \{i, j\}, y \in \{1, 2, 3\}$, and tripods L_i, L_j in G' with centers c_i, c_j such that

- for each $y \in \{1, 2, 3\}$, $\{v_{i,y}, v_{j,y}\} = \{\xi_{t_y}(i), \xi_{t_y}(j)\}$
- for each $m \in \{i, j\}$, L_m has feet $v_{m,1}, v_{m,2}, v_{m,3}$
- $L_i \cap L_j = c_i L_i v_{i,3} \cap c_j L_j v_{j,2}$ and it is a path that does not contain c_i, c_j.

We say that t_0 *has property C_{ij} in η* if there exist three pairwise disjoint paths R_i, R_j, R_{ij} and a path R in G' such that the ends of R_i are $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$, the ends of R_j are $\xi_{t_1}(j)$ and $\xi_{t_3}(j)$, the ends of R_{ij} are $\xi_{t_2}(j)$ and $\xi_{t_3}(i)$, and R is internally disjoint from R_i, R_j, R_{ij} and connects two of these three paths. We will denote these paths as $R_i(t_0), R_j(t_0), R_{ij}(t_0), R(t_0)$ when we want to emphasize they are in the η-torso at the major vertex t_0.

We say that the path P_i of a left or right t_0-linkage is *confined* if it is a subgraph of the η-torso at t_0.

Now let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) with orderings ξ_t and specified linkages. Let $t_0 \in V(T_h)$ be a major vertex with trinity (t_1, t_2, t_3), and let $P_1, P_2, \ldots, P_s$ be the left specified t_0-linkage. We define A_{t_0} to be the set of integers $i \in \{1, 2, \ldots, s\}$ such that the path P_i is confined, and we define B_{t_0} in the same way but using the right specified t_0-linkage instead. Define C_{t_0} as the set of all triples (i, l, m) such that $i \in \{1, 2, \ldots, s\}$, the path P_i is not confined and when following P_i from $\xi_{t_1}(i)$, it exits the η-torso at t_0 for the first time at $\xi_{t_3}(l)$ and re-enters the η-torso at t_0 for the last time at $\xi_{t_3}(m)$. Let D_{t_0} be defined similarly, but using the right t_0-linkage instead. We call the sets A_{t_0}, B_{t_0}, C_{t_0} and D_{t_0} the *confinement sets for η at t_0 with respect to the specified linkages.*

Let A_{t_0} and B_{t_0} be the confinement sets for η at t_0. We say that t_0 *has property C in η* if s is even, A_{t_0} and B_{t_0} are disjoint and both have size $s/2$, and there exist disjoint paths $R_1, R_2, \dots, R_{3s/2}$ in G' in such a way that

- each R_i is a subpath of both the left specified t_0-linkage and the right specified t_0-linkage,
- for $i \in A_{t_0}$, the path R_i has ends $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$,
- for $i \in B_{t_0}$ the path R_i has ends $\xi_{t_1}(i)$ and $\xi_{t_3}(i)$, and
- for $i = s+1, s+2, \dots, 3s/2$ the path R_i has one end $\xi_{t_2}(k)$ and the other and $\xi_{t_3}(l)$ for some $k \in B_{t_0}$ and $l \in A_{t_0}$.

Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be a cascade in (T, X) and let $\gamma : T_{h'} \hookrightarrow T_h$ be a monotone homeomorphic embedding. Then the composite mapping $\eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T$ is a cascade in (T, X) of height h', and we will call it a *subcascade of η.*

Lemma 3.3.3. *Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) with orderings ξ_t, specified linkages and common intersection set I, let $\gamma : T_{h'} \hookrightarrow T_h$ be a monotone homeomorphic embedding, and let $\eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T$ be a subcascade of η of height h'. Then for every major vertex $t_0 \in V(T_{h'})$*

(i) *η' is an ordered cascade with orderings $\xi_{\gamma(t)}$ and common intersection set I,*

(ii) *if the vertex $\gamma(t_0)$ has property A_{ij} (B_{ij}, C_{ij}, resp.) in η, then t_0 has property A_{ij} (B_{ij}, C_{ij}, resp.) in η'.*

Furthermore, the specified linkages for η' may be chosen in such a way that

(iii) $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0}) = (A_{\gamma(t_0)}, B_{\gamma(t_0)}, C_{\gamma(t_0)}, D_{\gamma(t_0)})$,

(iv) *the vertex t_0 has property C in η' if and only if $\gamma(t_0)$ has property C in η, and*

(v) *if the specified linkages for η are minimal, then the specified linkages for η' are minimal.*

Proof. For each major vertex $t \in V(T_{h'})$ or $t \in V(T_h)$ we denote its trinity by $(t_1(t), t_2(t), t_3(t))$. Assume t_0 is a major vertex of $T_{h'}$. Let $v_0 = \gamma(t_1(t_0)), v_1, \ldots, v_k = t_1(\gamma(t_0))$ be the minor vertices on $T_h[v_0, v_k]$. Let U be the union of the left (or right) linkage from $X_{\eta(v_i)} - I$ to $X_{\eta(v_{i+1})} - I$ for all $i \in \{0, 1, \ldots, k-1\}$ depending on whether v_{i+1} is a left (or right) neighbor of its parent. Let P be the left specified $\gamma(t_0)$-linkage and Q be the right specified $\gamma(t_0)$-linkage. Then $U \cup P$ is a left t_0-linkage and $U \cup Q$ is a right t_0-linkage. We designate $U \cup P$ to be the left specified t_0-linkage and $U \cup Q$ to be the right specified t_0-linkage. It is easy to see that this choice satisfies the conclusion of the lemma. $\square$

Let (T, X) be a tree-decomposition of a graph G, and let η be an ordered cascade with specified linkages in (T, X) of height h and size $|I| + s$, where I is the common intersection set. We say that η is *regular* if there exist sets $A, B \subseteq \{1, 2, \ldots, s\}$, and sets C and D such that the confinement sets A_{t_0}, B_{t_0}, C_{t_0} and D_{t_0} satisfy $A_{t_0} = A$, $B_{t_0} = B$, $C_{t_0} = C$ and $D_{t_0} = D$ for every major vertex $t_0 \in V(T_h)$.

Lemma 3.3.4. *For every two positive integers a and s there exists a positive integer $h = h(a, s)$ such that the following holds. Let (T, X) be a linked tree-decomposition of a graph G. If there exists an injective cascade η of height h in (T, X), then there exists a regular cascade $\eta' : T_a \hookrightarrow T$ of height a in (T, X) with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_a)$ such that η' has the same size and common intersection set as η.*

Proof. Let η be an injective cascade of size $|I| + s$ and height h in (T, X), where we will specify h in a moment. By Lemma 3.3.1 η can be turned into an ordered cascade with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_h)$. For every major vertex $t_0 \in V(T_h)$, the number of possible quadruples $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0})$ is a finite number $k = k(s)$ that depends only on s.

Consider each choice of $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0})$ as a color; then by Lemma 3.3.2, there exists a positive integer $h = h(a, k)$ such that there exists a monotone homeomorphic embedding $\gamma : T_a \hookrightarrow T_h$ such that the quadruple $(A_{\gamma(t)}, B_{\gamma(t)}, C_{\gamma(t)}, D_{\gamma(t)})$ for η is the same for every $t \in V(T_a)$. Now, let $\eta' = \eta \circ \gamma : T_a \to T$. Then η' is as desired by Lemma 3.3.3. $\square$

The following is the main result of this section.

Theorem 3.3.5. *For any two positive integers a and w, there exists a positive integer $p = p(a, w)$ such that the following holds. Let k be an integer such that $k \leq w$ and let G be a k-connected graph of tree-width less than w and path-width at least p. Then G has a tree-decomposition (T, X) such that:*

- *(T, X) has width less than w,*
- *(T, X) satisfies (W1)–(W7), and*
- *for some s, where $k \leq s \leq w$, there exists a regular cascade $\eta : T_a \hookrightarrow T$ of height a and size s in (T, X) with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_a)$.*

Proof. Given positive integers a and w let h be as in Lemma 3.3.4, and let $p = p(h, w)$ be as in Lemma 3.2.6. We claim that p satisfies the conclusion of the theorem. To see that let G be a graph of tree-width less than w and path-width at least p. By Theorem 2.1.4, G admits a tree-decomposition (T, X) of width less than w satisfying (W1)–(W7). By Lemma 3.2.6 there is an injective cascade of height h in (T, X). Let s be the size of this cascade, then

$s \leq w$. If G is k-connected, then $s \geq k$. The last conclusion of the theorem follows from Lemma 3.3.4. $\square$

3.4 Taming Linkages

Lemma 3.4.6, the main result of this section, states that there are essentially only two types of linkage.

Let $s > 0$ be an integer. Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) with size $|I|+s$ and orderings ξ_t, where I is the common intersection set of η. Let $t_0 \in V(T_h)$ be a major vertex, let (t_1, t_2, t_3) be the trinity at t_0, let G' be the η-torso at t_0, and let $i, j \in \{1, 2, \ldots, s\}$ be distinct. We say that t_0 *has property AB_{ij} in η* if there exist disjoint paths L_i, L_j and disjoint paths R_i, R_j in G' such that the two ends of L_m are $\xi_{t_1}(m)$ and $\xi_{t_2}(m)$ for each $m \in \{i, j\}$ and the two ends of R_m are $\xi_{t_1}(m)$ and $\xi_{t_3}(m)$ for each $m \in \{i, j\}$.

Lemma 3.4.1. *Let (T, X) be a tree-decomposition of a graph G. Let $\eta : T_1 \hookrightarrow T$ be an ordered cascade in (T, X) with orderings ξ_t of height one and size $s + |I|$, where I is the common intersection set. Let t_0 be the major vertex in T_1, and let $i, j \in \{1, 2, \ldots, s\}$ be distinct. If t_0 has property AB_{ij} in η, then t_0 has either property A_{ij} or property B_{ij} in η.*

Proof. Let (t_1, t_2, t_3) be the trinity at t_0. Let G' be the η-torso at t_0. Since t_0 has property AB_{ij} in η, there exist disjoint paths L_i, L_j and disjoint paths R_i, R_j in G' such that two endpoints of L_m are $\xi_{t_1}(m)$ and $\xi_{t_2}(m)$ for all $m \in \{i, j\}$, and two endpoints of R_m are $\xi_{t_1}(m)$ and $\xi_{t_3}(m)$ for all $m \in \{i, j\}$.

We may choose L_i, L_j, R_i, R_j such that $|E(L_i) \cup E(L_j) \cup E(R_i) \cup E(R_j)|$ is as small as possible.

Let $x_k = \xi_{t_1}(k)$ and $z_k = \xi_{t_3}(k)$ for $k \in \{i, j\}$. Starting from z_i, let a be the first vertex where R_i meets $L_i \cup L_j$, and starting from z_j, let b be the first vertex where R_j meets $L_i \cup L_j$. If a and b are not on the same path (one on L_i and the other on L_j), then by considering L_i, L_j and the parts of R_i and R_j from z_i to a and from z_j to b we see that t_0 has property A_{ij} in η.

If a and b are on the same path, then we may assume they are on L_i. We may also assume that $a \in L_i[y_i, b]$. Then following R_i from a away from z_i, the paths R_i and L_i eventually split; let c be the vertex where the split occurs. In other words, c is such that $aL_ic \cap aR_ic$ is a path and its length is maximum. Let d be the first vertex on $cR_ix_i \cap (L_i \cup L_j) - \{c\}$ when traveling on R_i from c to x_i. If $d \in V(L_i)$, then by replacing cL_id by cR_id we obtain a contradiction to the choice of L_i, L_j, R_i, R_j. Thus $d \in V(L_j)$. Now L_i, L_j and the paths z_iR_id and z_jR_jb show that t_0 has property B_{ij} in η. $\square$

Let (T, X) be a tree-decomposition of a graph G and let $\eta : T_h \hookrightarrow T$ be an injective cascade in (T, X) of height h and size $|I| + s$, where I is the common intersection set. Let v be a vertex of T_h and let Y consist of $\eta(v)$ and the vertex-sets of all components of $T - \eta(v)$ that do not contain the image under η of the minor root of T_h. Let H be the subgraph of G induced by $\bigcup_{t \in Y} X_t - I$. We will call H the *outer graph at* v.

Lemma 3.4.2. *Let (T, X) be a tree-decomposition satisfying (W6) of a graph G and let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) of height h and size $|I| + s$, where I is the common intersection set. Let v_0 be a major vertex of T_h and let v be a minor vertex adjacent to v_0. let Y consist of $\eta(v)$ and the vertex-set of the component of $T - \eta(v)$ that contains $\eta(v_0)$. Let H be the subgraph of G induced by $\bigcup_{t \in Y} X_t - I$. Let $x, y \in X_{\eta(v)}$. Then there exists a path of length at least two with ends x and y and every internal vertex in $V(H) - X_{\eta(v)}$. In particular, let v be a minor vertex of T_h at height at most $h - 1$, let H be the outer graph at v, and let $x, y \in X_{\eta(v)}$. Then there exists a path of length at least two with ends x and y and every internal vertex in $V(H) - X_{\eta(v)}$.*

Proof. Let v_1 be a child of v_0 if v is the parent of v_0, otherwise let v_1 be the parent of v_0. Let B be the component of $T - \eta(v)$ that contains $\eta(v_1)$. We show that x is B-tied. This is obvious if $x \in I$, and so we may assume that $x \notin I$. Since η is ordered, there exist

s disjoint paths from $X_{\eta(v)} - I$ to $X_{\eta(v_1)} - I$ in $G - I$. It follows that each of the paths uses exactly one vertex of $X_{\eta(v)} - I$, and that vertex is its end. Let P be the one of those paths that ends in x, and let x' be the neighbor of x in P. The vertex x' exists, because $X_{\eta(v)} \cap X_{\eta(v_1)} = I$. By (W1) there exists a vertex $t \in V(T)$ such that $x, x' \in X_t$. Since $P - x$ is disjoint from $X_{\eta(v)}$, it follows from Lemma 2.1.1 applied to the path $P - x$ and vertices t and $\eta(v_1)$ of T that $t \in V(B)$. Thus x is B-tied and the same argument shows that so is y. Hence the lemma follows from (W6). □

We will refer to a path as in Lemma 3.4.2 as a W6-*path*.

Let h, h' be integers. We say that a homeomorphic embedding $\gamma : T_{h'} \hookrightarrow T_h$ is *weakly monotone* if for every two vertices $t, t' \in V(T_{h'})$

- if t' is a descendant of t in $T_{h'}$, then the vertex $\gamma(t')$ is a descendant of $\gamma(t)$ in T_h
- if t is a minor vertex of $T_{h'}$, then the vertex $\gamma(t)$ is minor in T_h.

Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be a cascade in (T, X) and let $\gamma : T_{h'} \hookrightarrow T_h$ be a weakly monotone homeomorphic embedding. Then the composite mapping $\eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T$ is a cascade in (T, X) of height h', and we will call it a *weak subcascade of* η.

Lemma 3.4.3. *Let $s \geq 2$ be an integer, let (T, X) be a tree-decomposition of a graph G satisfying (W6), and let $\eta : T_5 \hookrightarrow T$ be a regular cascade in (T, X) of height five and size $|I| + s$ with specified linkages that are minimal, where I is the common intersection set of η. Then either there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one such that the unique major vertex of T_1 has property A_{ij} or B_{ij} in η' for some distinct integers $i, j \in \{1, 2, \ldots, s\}$, or the major root of T_5 has property C in η.*

Proof. We will either construct a weakly monotone homeomorphic embedding $\gamma : T_1 \hookrightarrow T_5$ such that in $\eta' = \eta \circ \gamma$ the major root of T_1 will have property AB_{ij} for some distinct $i, j \in \{1, 2, ..., s\}$, or establish that the major root of T_5 has property C in η. By Lemma 3.4.1 this will suffice.

Since η is regular, there exist sets A, B, C, D as in the definition of regular cascade. Let t_0 be the unique major vertex of T_1 and let (t_1, t_2, t_3) be its trinity. Let u_0 be the major root of T_5 and let (v_1, v_2, v_3) be its trinity. Let u_1 be the major vertex of T_5 of height one that is adjacent to v_3 and let (v_3, v_4, v_5) be its trinity. Let us recall that for a major vertex u of T_5 we denote the paths in the specified left u-linkage by $P_i(u)$ and the paths in the specified right u-linkage by $Q_i(u)$. If there exist two distinct integers $i, j \in A \cap B$, then the paths $P_i(u_0), P_j(u_0), Q_i(u_0), Q_j(u_0)$ show that u_0 has property AB_{ij} in η. Let $\gamma : T_1 \hookrightarrow T_5$ be the homeomorphic embedding that maps t_0, t_1, t_2, t_3 to u_0, v_1, v_2, v_3, respectively. Then $\eta' = \eta \circ \gamma$ is as desired. We may therefore assume that $|A \cap B| \leq 1$.

For $i \in \{1, 2, \ldots, s\} - A$ the path $P_i(u_0)$ exits and re-enters the η-torso at u_0, and it does so through two distinct vertices of $X_{\eta(v_3)}$. But $|X_{\eta(v_3)} \backslash I| = s$, and hence $|A| \geq s/2$. Similarly $|B| \geq s/2$. By symmetry we may assume that $|B| \geq |A|$. It follows that $|A| = \lceil s/2 \rceil$, and hence for $i \in \{1, 2, \ldots, s\} - A$ and every major vertex w of T_5 the path $P_i(w)$ exits and re-enters the η-torso at w exactly once. The set C includes an element of the form (i, l, m), which means that the vertices $\xi_{w_1}(i), \xi_{w_3}(l), \xi_{w_3}(m), \xi_{w_2}(i)$ appear on the path $P_i(w)$ in the order listed. Let $l_i := l, m_i := m$, $x_i(w) := \xi_{w_3}(l)$, $y_i(w) := \xi_{w_3}(m)$, $X_i(w) := \xi_{w_1}(i) P_i(w) x_i(w)$ and $Y_i(w) := y_i(w) P_i(w) \xi_{w_2}(i)$. Thus $X_i(w)$ and $Y_i(w)$ are subpaths of the η-torso at w. We distinguish two main cases.

Main case 1: $|A \cap B| = 1$. Let j be the unique element of $A \cap B$. We claim that $B - A \neq \emptyset$. To prove the claim suppose for a contradiction that $B \subseteq A$. Thus $|B| = 1$, and since $|B| \geq |A|$ we have $|A| = 1$, and hence $s = 2$. We may assume, for the duration of this paragraph, that $A = B = \{1\}$. The paths $P_1(u_0), X_2(u_0), Y_2(u_0)$ are pairwise disjoint, because they are subgraphs of the specified left u_0-linkage. The path $Q_2(u_0)$ is unconfined, and hence it has a subpath R joining $\xi_{v_2}(1)$ and $\xi_{v_2}(2)$ in the outer graph at v_2. It follows that $P_1(u_0) \cup R \cup Y_2(u_0)$ and $X_2(u_0)$ are disjoint paths from $X_{\eta(v_1)}$ to $X_{\eta(v_3)}$, and it follows from the minimality of the specified u_0-linkage that they form the specified right u_0-linkage, contrary to $1 \in A$. This proves the claim that $B - A \neq \emptyset$, and so we may select an element $i \in B - A$.

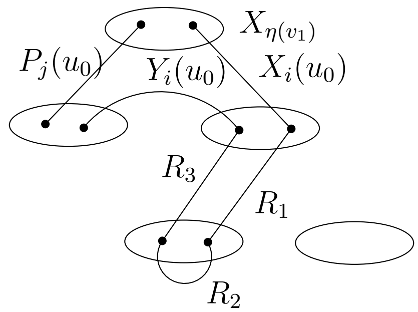


Figure 3.1: First case of the construction of the path R.

Let us assume as a case that either $l_i \in A$ or $l_i \notin B$. In this case we let γ map t_0, t_1, t_2, t_3 to u_0, v_1, v_2, v_5, respectively, and we will prove that t_0 has property AB_{ij} in η'. To that end we need to construct two pairs of disjoint paths. The first pair is $Q_i(u_0) \cup Q_i(u_1)$ and $Q_j(u_0) \cup Q_j(u_1)$. The second pair will consist of $P_j(u_0)$ and another path from $\xi_{v_1}(i)$ to $\xi_{v_2}(i)$ which is a subgraph of a walk that we are about to construct. It will consist of $X_i(u_0) \cup Y_i(u_0)$ and a walk R in the outer graph of v_3 with ends $x_i(u_0)$ and $y_i(u_0)$. To construct the walk R we will construct paths R_1, R_2 and a walk R_3, whose union will contain the desired walk R. If $l_i \in A$, then we let $R_1 := P_{l_i}(u_1)$. If $l_i \notin B$, then the path $Q_{l_i}(u_1)$ is unconfined, and hence includes a subpath R_1 from $x_i(u_0)$ to $X_{\eta(v_4)}$ that is a subgraph of the η-torso at u_1. We need to distinguish two subcases depending on whether $m_i \in B$. Assume first that $m_i \notin B$ and refer to Figure 3.1. Then similarly as above the path $Q_{m_i}(u_1)$ is unconfined, and hence includes a subpath R_3 from $y_i(u_0)$ to $X_{\eta(v_4)}$ that is a subgraph of the η-torso at u_1, and we let R_2 be a W6-path in the outer graph at v_4 joining the ends of R_1 and R_3 in $X_{\eta(v_4)}$. This completes the subcase $m_i \notin B$, and so we may assume that $m_i \in B$. In this subcase we define $R_3 := Y_i(u_1) \cup Q_{m_i}(u_1)$ and we define R_2 as above. See Figure 3.2. This completes the case that either $l_i \in A$ or $l_i \notin B$.

Next we consider the case $l_i \in B$ and $m_i \notin A - B$. We proceed similarly as in the previous paragraph, but with these exceptions: the homeomorphic embedding γ will map t_3 to v_4, rather than v_5, the first pair of disjoint paths will now be $Q_i(u_0) \cup P_i(u_1)$

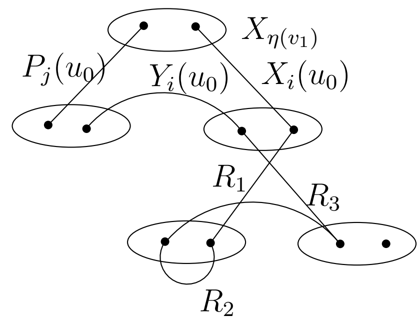


Figure 3.2: Second case of the construction of the path R.

and $Q_j(u_0) \cup P_j(u_1)$, and for the second pair we define $R_1 = Q_{l_i}(u_1)$, $R_3 = X_{m_i}(u_1)$ if $m_i \notin A$ and $R_3 = Q_{m_i}(u_1)$ if $m_i \in B$, and R_2 will be a W6-path in the outer graph of v_5 joining the ends of R_1 and R_3.

Therefore assume that $l_i \in B - A$ and $m_i \in A - B$ for every $i \in B - A$. Let u_2 be the major vertex of T_5 at height two whose trinity includes v_5 and assume its trinity is (v_5, v_6, v_7). Let u_3 be the major vertex of T_5 at height three whose trinity includes v_7 and assume its trinity is (v_7, v_8, v_9). Let γ map t_0, t_1, t_2, t_3 to u_0, v_1, v_2, v_8, respectively. Then t_0 also has property AB_{ij} in η'. To see that the first pair of disjoint paths is $Q_i(u_0) \cup Q_i(u_1) \cup Q_i(u_2) \cup P_i(u_3)$ and $Q_j(u_0) \cup Q_j(u_1) \cup Q_j(u_2) \cup P_j(u_3)$. The first path of the second pair is $P_j(u_0)$. Let $R_1 = Y_i(u_0) \cup Q_{m_i}(u_1) \cup P_{m_i}(u_2)$, $R_2 = P_j(u_2) \cup Q_j(u_2) \cup Q_j(u_3)$, and $R_3 = X_i(u_0) \cup Q_{l_i}(u_1) \cup X_{l_i}(u_2) \cup X_{l_{l_i}}(u_3)$. Then the second path of the second pair is a path from $\xi_{v_1}(i)$ to $\xi_{v_2}(i)$ that is a subgraph of $R_1 \cup R_2 \cup R_3 \cup R_4 \cup R_5$, where R_4 is a W6-path in the outer graph of v_6 joining the ends of R_1 and R_2, and R_5 is a W6-path in the outer graph of v_9 joining the ends of R_2 and R_3. See Figure 3.3. This completes main case 1.

Main case 2: $A \cap B = \emptyset$. It follows that s is even and $|A| = |B| = s/2$. Assume as a case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$. But the integers l_i, m_i are pairwise distinct, and so if $l_i, m_i \in A$, then there exists $j \in B$ such that $l_j, m_j \in B$, and similarly if $l_i, m_i \in B$. We may therefore assume that $l_i, m_i \in A$ and $l_j, m_j \in B$ for some distinct $i, j \in B$. Let us recall that u_2 is the child of v_5 and (v_5, v_6, v_7) is its trinity. We

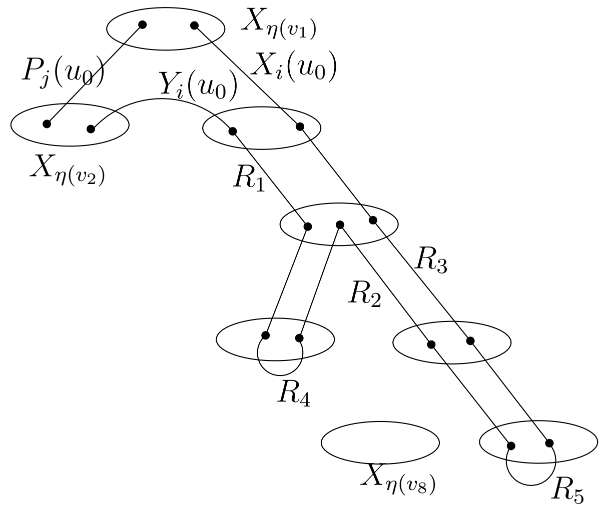


Figure 3.3: Second pair when $l_i \in B - A$ and and $m_i \in A - B$.

let γ map t_0, t_1, t_2, t_3 to u_0, v_1, v_2, v_6, respectively, and we will prove that t_0 has property AB_{ij} in η'. To that end we need to construct two pairs of disjoint paths. The first pair is $Q_i(u_0) \cap Q_i(u_1) \cap P_i(u_2)$ and $Q_j(u_0) \cap Q_j(u_1) \cap P_j(u_2)$. The first path of the second pair will consist of the union of $X_i(u_0)$ with a subpath of $Q_{l_i}(u_1)$ from $X_{\eta(v_3)}$ to $X_{\eta(v_4)}$, and $Y_i(u_0)$ with a subpath of $Q_{m_i}(u_1)$ from $X_{\eta(v_3)}$ to $X_{\eta(v_4)}$, and a suitable W6-path in the outer graph of v_4 joining their ends, and the second path will consist of the union of $X_j(u_0) \cup Q_{l_j}(u_1) \cup Q_{l_j}(u_2)$ and $Y_j(u_0) \cup Q_{m_j}(u_1) \cup Q_{m_j}(u_2)$ and a suitable W6-path in the outer a graph of v_7 joining their ends. See Figure 3.4. This completes the case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$.

We may therefore assume that for every $i \in B$ one of l_i, m_i belongs to A and the other belongs to B. Let us recall that for every $i \in B$ a subpath of $P_i(u_0)$ joins $\xi_{v_3}(l_i)$ to $\xi_{v_3}(m_i)$ in the outer graph at v_3 and is disjoint from the η-torso at u_0, except for its ends. Let J be the union of these subpaths; then J is a linkage from $\{\xi_{v_3}(i) : i \in A\}$ to $\{\xi_{v_3}(i) : i \in B\}$. For $i \in B$ the path $Q_i(u_0)$ is a subgraph of the η-torso at u_0. For $i \in A$ the intersection of the path $Q_i(u_0)$ with the η-torso at u_0 consists of two paths, one from $X_{\eta(v_1)}$ to $X_{\eta(v_2)}$, and the other from $X_{\eta(v_2)}$ to $X_{\eta(v_3)}$. Let L denote the union of these subpaths over all $i \in A$. It follows that $J \cup L \cup \bigcup_{i \in B} Q_i(u_0)$ is a linkage from $X_{\eta(v_1)}$ to $X_{\eta(v_2)}$, and so by

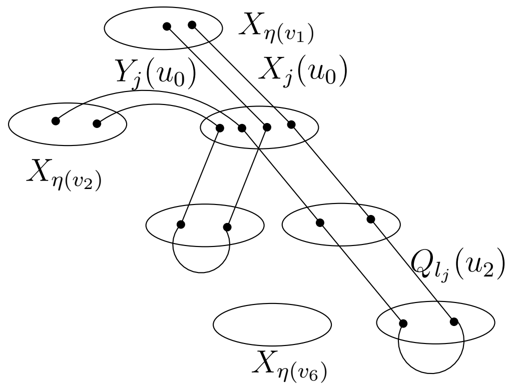


Figure 3.4: Second pair when $l_i, m_i \in A$ and $l_j, m_j \in B$ for some distinct $i, j \in B$.

the minimality of the specified u_0-linkages, it is equal to the specified left u_0-linkage. It follows that u_0 has property C in η. $\square$

Lemma 3.4.4. *Let (T, X) be a tree-decomposition of a graph G satisfying (W6) and (W7). If there exists a regular cascade $\eta : T_3 \hookrightarrow T$ with orderings ξ_t in which every major vertex has property C, then there is a weak subcascade η' of η of height one such that the major vertex in η' has property C_{ij} for some i, j.*

Proof. Let the common confinement sets for η be A, B, C, D. For a major vertex $w \in V(T_3)$ with trinity (v_1, v_2, v_3) there are disjoint paths in the η-torso at w as in the definition of property C. For $a \in A$ and $b \in B$ let $R_a(w)$ denote the path with ends $\xi_{v_1}(a)$ and $\xi_{v_2}(a)$, let $R_b(w)$ denote the path with ends $\xi_{v_1}(b)$ and $\xi_{v_3}(b)$, and let $R_{ab}(w)$ denote the path with ends $\xi_{v_2}(b)$ and $\xi_{v_3}(a)$.

Assume the major root of T_3 is u_0 and its trinity is (v_1, v_2, v_3), and let I be the common intersection set of η. Then $\eta(v_1), \eta(v_2), \eta(v_3)$ is a triad in T with center $\eta(u_0)$ and for all $i \in \{1, 2, 3\}$ we have $X_{\eta(v_i)} \cap X_{\eta(u_0)} = I = X_{\eta(v_1)} \cap X_{\eta(v_2)} \cap X_{\eta(v_3)}$, and hence the triad is not X-separable by (W7). Thus by Lemma 2.1.1 there is a path $R(u_0)$ connecting two of the three sets of disjoint paths in the η-torso at u_0. Assume without loss of generality that one end of $R(u_0)$ is in a path $R_i(u_0)$, where $i \in A$. Then the other end of $R(u_0)$ is either in a path $R_j(u_0)$, where $j \in B$; or in a path $R_{aj}(u_0)$, where $j \in B$ and $a \in A$. In the former case we define $a \in A$ to be such that $R_{aj}(u_0)$ is a path in the family.

Let the major root of T_1 be t_0 and its trinity be (t_1, t_2, t_3). Let $\gamma(t_0) = u_0$, $\gamma(t_1) = v_1$, $\gamma(t_2) = v_2$. Let the major vertex that is the child of v_3 be u_1, and the trinity at u_1 be (v_3, v_4, v_5). Let $\gamma(t_3) = v_5$. We will prove that t_0 has property C_{ij} in $\eta' = \eta \circ \gamma$. Let $b \in B$ be such that $R_{ib}(u_1)$ is a member of the family of the disjoint paths in the η-torso at u_1 as in the definition of property C. By Lemma 3.4.2, there exists a $W6$-path P in the outer graph at v_4 joining $\xi_{v_4}(a)$ and $\xi_{v_4}(b)$. By considering the paths $R_i(u_0)$, $R_j(u_0) \cup R_j(u_1)$, $R_{aj}(u_0) \cup R_a(u_1) \cup P \cup R_{ib}(u_1)$ and $R(u_0)$ we find that t_0 has property C_{ij} in η', as desired. $\square$

Lemma 3.4.5. *Let $s \geq 2$ be an integer and let (T, X) be a tree-decomposition of a graph G satisfying (W6). Let $\eta : T_3 \hookrightarrow T$ be an ordered cascade in (T, X) of height three and size $|I| + s$ with orderings ξ_t and common intersection set I such that every major vertex of T_3 has property C_{ij} for some distinct $i, j \in \{1, 2, \ldots, s\}$. Then there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one such that the unique major vertex of T_1 has property B_{ij} in η'.*

Proof. Assume that three major vertices at height zero and one of T_3 are u_0, u_1, u_2. Let the trinity at u_0 be (v_1, v_2, v_3), the trinity at u_1 be (v_2, v_4, v_5), and the trinity at u_2 be (v_3, v_6, v_7). Assume the major vertex of T_1 is t_0, and its trinity is (t_1, t_2, t_3). For a major vertex $w \in V(T_3)$ let $R_i(w), R_j(w), R_{ij}(w)$ and $R(w)$ be as in the definition of property C_{ij}.

We need to find a weakly monotone homeomorphic embedding $\gamma : T_1 \hookrightarrow T_3$ such that $\eta' = \eta \circ \gamma$ satisfies the requirement. Set $\gamma(t_0) = u_0$ and $\gamma(t_1) = v_1$. Our choice for $\gamma(t_2)$ will be v_4 or v_5, depending on which two of the three paths $R_i(u_1), R_j(u_1), R_{ij}(u_1)$ in the η-torso at u_1 the path $R(u_1)$ is connecting. If $R(u_1)$ is between $R_i(u_1)$ and $R_j(u_1)$, then choose either v_4 or v_5 for $\gamma(t_2)$. If $R(u_1)$ is between $R_i(u_1)$ and $R_{ij}(u_1)$, then set $\gamma(t_2) = v_4$, and if it is between $R_j(u_1)$ and $R_{ij}(u_1)$, then set $\gamma(t_2) = v_5$. Do this similarly for $\gamma(t_3)$. Then $\eta' = \eta \circ \gamma$ will satisfy the requirement. In fact, we will prove this for the case when $R(u_1)$ is between $R_i(u_1)$ and $R_{ij}(u_1)$ and $R(u_2)$ is between $R_j(u_2)$ and $R_{ij}(u_2)$. See Figure 3.5. The other five cases are similar.

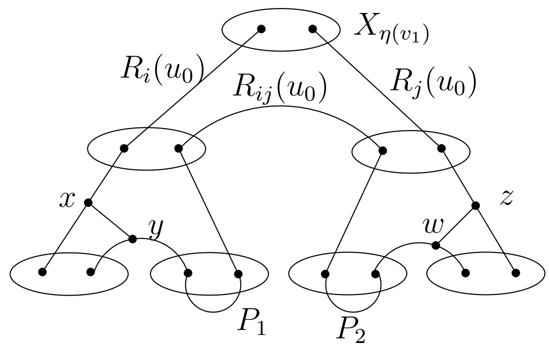


Figure 3.5: The case when $R(u_1)$ is between $R_i(u_1)$ and $R_{ij}(u_1)$ and $R(u_2)$ is between $R_j(u_2)$ and $R_{ij}(u_2)$.

In this case, our choice is $\gamma(t_0) = u_0, \gamma(t_1) = v_1, \gamma(t_2) = v_4, \gamma(t_3) = v_7$. Assume the two endpoints of $R(u_1)$ are x and y and the two endpoints of $R(u_2)$ are w and z. By Lemma 3.4.2, there exists a W6-path P_1 between $\xi_{v_5}(i)$ and $\xi_{v_5}(j)$ in the outer graph at v_5 and a W6-path P_2 between $\xi_{v_6}(i)$ and $\xi_{v_6}(j)$ in the outer graph at v_6. Now let

$$P = yR_{ij}(u_1)\xi_{v_5}(i) \cup P_1 \cup R_j(u_1) \cup R_{ij}(u_0) \cup R_i(u_2) \cup P_2 \cup \xi_{v_6}(j)R_{ij}(u_2)w,$$

$$L_i = R_i(u_0) \cup R_i(u_1) \cup R(u_1) \cup P \cup wR_{ij}(u_2)\xi_{v_7}(i)$$

and

$$L_j = R_j(u_0) \cup R_j(u_2) \cup R(u_2) \cup P \cup yR_{ij}(u_1)\xi_{v_4}(j).$$

The tripods L_i and L_j show that the major vertex of $\eta' = \eta \circ \gamma : T_1 \hookrightarrow T$ has property B_{ij}.

$\square$

Lemma 3.4.6. *For every positive integers h' and $w \geq 2$ there exists a positive integer $h = h(h', w)$ such that the following holds. Let s be a positive integer such that $2 \leq s \leq w$. Let (T, X) be a tree-decomposition of a graph G of width less than w and satisfying (W6) and (W7). Assume there exists a regular cascade $\eta : T_h \hookrightarrow T$ of size $|I| + s$ with specified* *linkages that are minimal, where I is its common intersection set. Then there exist distinct integers $i, j \in \{1, 2, \ldots, s\}$ and a weak subcascade $\eta' : T_{h'} \hookrightarrow T$ of η of height h' such that*

- *every major vertex of $T_{h'}$ has property A_{ij} in η', or*
- *every major vertex of $T_{h'}$ has property B_{ij} in η'*

Proof. Let $h(a, k)$ be the function of Lemma 3.3.2, let $a_3 = 3h'$, $a_2 = h(a_3, 2\binom{w}{2})$, $a_1 = 5a_2$ and $h = h(a_1, 2)$. Consider having property C or not having property C as colors, then by Lemma 3.3.2 there exists a monotone homeomorphic embedding $\gamma : T_{a_1} \hookrightarrow T_h$ such that either $\gamma(t)$ has property C in η for every major vertex $t \in V(T_{a_1})$ or $\gamma(t)$ does not have property C in η for every major vertex $t \in V(T_{a_1})$. By Lemma 3.3.3 $\eta_1 = \eta \circ \gamma : T_{a_1} \hookrightarrow T$ is still a regular cascade with specified linkages that are minimal. Also, either t has property C in η_1 for every major vertex $t \in V(T_{a_1})$ or t does not have property C in η_1 for every major vertex $t \in V(T_{a_1})$.

If t has property C in η_1 for every major vertex $t \in V(T_{a_1})$, then by Lemma 3.4.4 there exists a weak subcascade η_2 of η_1 of height a_2 such that every major vertex of T_{a_2} has property C_{ij} in η_2 for some distinct $i, j \in \{1, 2, ..., s\}$. Consider each choice of pair i, j as a color; then by Lemma 3.3.2 there exists a monotone homeomorphic embedding $\gamma_1 : T_{a_3} \hookrightarrow T_{a_2}$ such that for some distinct $i, j \in \{1, 2, ..., s\}$, $\gamma_1(t)$ has property C_{ij} in η_2 for every major vertex $t \in V(T_{a_3})$. Let $\eta_3 = \eta_2 \circ \gamma_1$. Then by Lemma 3.3.3 this implies t has property C_{ij} in η_3 for every major vertex $t \in V(T_{a_3})$. Then by Lemma 3.4.5 there exists a weak subcascade $\eta_4 : h' \hookrightarrow a_3$ of η_3 such that every major vertex of $T_{h'}$ has property B_{ij} in η_4. Hence η_4 is as desired.

If t does not have property C in η_1 for every major vertex $t \in V(T_{a_1})$, then by Lemma 3.4.3 there exists a weak subcascade η_2 of η_1 of height a_2 such that every major vertex of T_{a_2} has property A_{ij} or B_{ij} for some distinct $i, j \in \{1, 2, ..., s\}$. Consider each property A_{ij} or B_{ij} as a color; then by Lemma 3.3.2 there exists a monotone homeomorphic embedding

$\gamma_1 : T_{h'} \hookrightarrow T_{a_2}$ such that for some distinct $i, j \in \{1, 2, ..., s\}$, either $\gamma_1(t)$ has property A_{ij} in η_2 for every major vertex $t \in V(T_{h'})$ or $\gamma_1(t)$ has property B_{ij} in η_2 for every major vertex $t \in V(T_{h'})$. Let $\eta_3 = \eta_2 \circ \gamma_1$. Then t has property A_{ij} in η_3 for every major vertex $t \in V(T_{h'})$ or t has property B_{ij} in η_3 for every major vertex $t \in V(T_{h'})$ by Lemma 3.3.3. Hence η_3 is as desired. $\square$

3.5 Proof of Theorem 1.1.3

By Lemmas 3.1.2 and 3.1.4 Theorem 1.1.3 is equivalent to the following theorem.

Theorem 3.5.1. *For any positive integer k, there exists a positive integer $p = p(k)$ such that for every 2-connected graph G, if G has path-width at least p, then G has a minor isomorphic to $\mathcal{P}_k$ or $\mathcal{Q}_k$.*

We need the following lemma.

Lemma 3.5.2. *Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) with orderings ξ_t of height h and size $s + I$, where I is the common intersection set, and let $i, j \in \{1, 2, \ldots, s\}$ be distinct and such that every major vertex of T_h has property B_{ij} in η. Let t be the minor root of T_h, and let w_1w_2 be the base edge of $\mathcal{Q}_h$. For every major vertex t_0 in T_h let V_{t_0} be the vertex set of the η-torso at t_0. Let G' be the subgraph of G induced by $\bigcup_{t_0} V_{t_0}$, where the union is taken over all major vertices $t_0 \in V(T_h)$. Then G' has a minor isomorphic to $\mathcal{Q}_h - w_1w_2$ in such a way $\xi_t(i)$ belongs to the node of w_1, $\xi_t(j)$ belongs to the node of w_2, and the node of each leaf of $\mathcal{Q}_h$ contains $\xi_r(i)$ or $\xi_r(j)$ for some minor vertex r at height h of T_h.*

Proof. We proceed by induction on h. Let t_0 be the major root of T_h, let (t_1, t_2, t_3) be its trinity, and let L_i and L_j be the tripods in the η-torso at t_0 as in the definition of property B_{ij}. The graph $L_i \cup L_j$ contains a path P joining $\xi_{t_1}(i)$ to $\xi_{t_1}(j)$ and a path P' joining $\xi_{t_2}(i)$ or $\xi_{t_2}(j)$ to P such that $\xi_{t_1}(i), \xi_{t_1}(j) \notin V(P')$, which shows that the lemma holds for $h = 1$.

We may therefore assume that $h > 1$ and that the lemma holds for $h-1$. For $k \in \{2,3\}$ let R_k be the subtree of T_h rooted at t_k, let η_k be the restriction of η to R_k, and let G_k be the subgraph of G induced by $\bigcup_{t_0} V_{t_0}$, where the union is taken over all major vertices $t_0 \in V(R_k)$. By the induction hypothesis applied to η_k and G_k, the graph G_k has a minor isomorphic to $\mathcal{Q}_{h-1} - u_1u_2$ in such a way $\xi_{t_k}(i)$ belongs to the node of u_1, $\xi_{t_k}(j)$ belongs to the node of u_2, where u_1u_2 is the base edge of $\mathcal{Q}_{h-1}$, and the node of each leaf of $\mathcal{Q}_{h-1}$ contains $\xi_r(i)$ or $\xi_r(j)$ for some minor vertex r at height $h-1$ of R_k. By using these two minors, the path P and the rest of the triads L_i and L_j we find that G' has the desired minor. □

Lemma 3.5.3. *For every two positive integers k and $w \geq 2$ there exists an integer h such that the following holds. Let (T, X) be a tree-decomposition of a graph G of width less than w and satisfying (W1)–(W7). Assume there exists a regular cascade $\eta : T_h \hookrightarrow T$ of size $|I| + s$ with specified linkages that are minimal, where I is its common intersection set and $2 \leq s \leq w$.*

(i) Then G has a minor isomorphic to $\mathcal{P}_k$ or $\mathcal{Q}_k$.

(ii) If $|I| \geq 1$, then G has a minor isomorphic to $\mathcal{P}'_k$ or $\mathcal{Q}'_k$.

(iii) If $|I| \geq 2$, then G has a minor isomorphic to $\mathcal{P}''_k$ or $\mathcal{Q}''_k$.

Proof. Let $h' = 4k + 1$, and let $h = h(h', w)$ be the number as in Lemma 3.4.6. By Lemma 3.4.6 there exist distinct integers $i, j \in \{1, 2, \ldots, s\}$ and a weak subcascade $\eta' : T_{h'} \hookrightarrow T$ of η of height h' such that

- every major vertex of $T_{h'}$ has property A_{ij} in η', or
- every major vertex of $T_{h'}$ has property B_{ij} in η'

Assume that every major vertex of $T_{h'}$ has property A_{ij} in η', and let R be the union of the corresponding tripods, over all major vertices $t \in V(T_{h'})$ at height at most $h'-2$. It follows that R is the union of two disjoint trees, each containing a subtree isomorphic to $T_{(h'-1)/2}$. Let t be a minor vertex of $T_{h'}$ at height $h'-1$. By Lemma 3.4.2 there exists a W6-path with ends $\xi_t(i)$ and $\xi_t(j)$ in the outer graph at t. Let R_1 be the union of these W6-paths for all minor vertices t at height $h'-1$. By contracting one of the trees comprising R and by considering R_1 we deduce that G has a $\mathcal{P}_k$ minor, as desired. If $|I| \geq 1$, assume $x \in I$. By Lemma 3.4.2 there exists a W6-path with ends x and $\xi_t(j)$ in the outer graph at t. Let R_2 be the union of these W6-paths for all minor vertices t at height $h'-1$. By contracting the tree that contains $\xi_t(i)$ of R and by considering R_1, R_2 we deduce that G has a $\mathcal{P}'_k$ minor, as desired. If $|I| \geq 2$, assume $x_1, x_2 \in I$. By Lemma 3.4.2 there exist a W6-path with ends x_1 and $\xi_t(j)$ and a W6-path with ends x_2 and $\xi_t(j)$ in the outer graph at t. Let R_3 be the union of these W6-paths for all minor vertices t at height $h'-1$. By contracting the tree that contains $\xi_t(i)$ of R and by considering R_1, R_3 we deduce that G has a $\mathcal{P}''_k$ minor, as desired.

We may therefore assume that every major vertex of $T_{h'}$ has property B_{ij} in η'. For every major vertex t_0 in $T_{h'}$ let V_{t_0} be the vertex set of the η-torso at t_0. Let G' be the subgraph of G induced by $\bigcup_{t_0} V_{t_0}$, where the union is taken over all major vertices $t_0 \in V(T_h)$ at height at most $h'-2$. It follows from Lemma 3.5.2 applied to T_{h-1} that G' has a minor isomorphic to $\mathcal{Q}_{h'-2}$, as desired. Let t be a minor vertex of $T_{h'}$ at height $h'-1$. If $|I| \geq 1$, assume $x \in I$. By Lemma 3.4.2 there exist a W6-path with ends x and $\xi_t(i)$ and a W6-path with ends x and $\xi_t(j)$ in the outer graph at t. Let R_1 be the union of these W6-paths for all minor vertices t at height $h'-1$. By considering R_1 and the minor isomorphic to $\mathcal{Q}_{h'-2}$ in G' we deduce that G has a $\mathcal{Q}'_{h'-2}$ minor, as desired. If $|I| \geq 2$, assume $x_1, x_2 \in I$. By Lemma 3.4.2 there exist W6-paths with ends a and b for all $a \in \{x_1, x_2\}$ and $b \in \{\xi_t(i), \xi_t(j)\}$ in the outer graph at t. Let R_2 be the union of these W6-paths for all minor vertices t at height $h'-1$. By considering R_2 and the minor isomorphic to $\mathcal{Q}_{h'-2}$ in G' we deduce that G has a $\mathcal{Q}''_{h'-2}$ minor, as desired. □

We deduce Theorem 3.5.1 from the following lemma.

Lemma 3.5.4. *Let k and w be positive integers. There exists an integer $p = p(k, w)$ such that for every 2-connected graph G, if G has tree-width less than w and path-width at least p, then G has a minor isomorphic to $\mathcal{P}_k$ or $\mathcal{Q}_k$.*

Proof. Let h' be as in Lemma 3.5.3 and let $h = \max\{h', k + 1\}$. Let p be as in Theorem 3.3.5 applied to $a = h$ and w. We claim that p satisfies the conclusion of the lemma. By Theorem 3.3.5, there exists a tree-decomposition (T, X) of G such that:

- (T, X) has width less than w,
- (T, X) satisfies (W1)–(W7), and
- for some s, where $2 \leq s \leq w$, there exists a regular cascade $\eta : T_h \hookrightarrow T$ of height h and size s in (T, X) with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_h)$.

Let I be the common intersection set of η, let ξ_t be the orderings, and let $s_1 = s - |I|$. Then $s_1 \geq 1$ by the definition of injective cascade.

Assume first that $s_1 = 1$. Since $s \geq 2$, it follows that $I \neq \emptyset$. Let $x \in I$. Let R be the union of the left and right specified t-linkage with respect to η, over all major vertices $t \in V(T_h)$ at height at most $h - 2$. The minimality of the specified linkages implies that R is isomorphic to a subdivision of T_{h-1}. Let t be a minor vertex of T_h at height $h - 1$. By Lemma 3.4.2 there exists a W6-path with ends $\xi_t(1)$ and x and every internal vertex in the outer graph at t. The union of R and these W6-paths shows that G has a $\mathcal{P}_k$ minor, as desired.

We may therefore assume that $s_1 \geq 2$. By Lemma 3.5.3(i), G has a minor isomorphic to $\mathcal{P}_k$ or $\mathcal{Q}_k$. $\square$

Proof of Theorem 3.5.1. Let a positive integer k be given. By Theorem 1.1.1 there exists an integer w such that every graph of tree-width at least w has a minor isomorphic to $\mathcal{P}_k$. Let $p = p(k, w)$ be as in Lemma 3.5.4. We claim that p satisfies the conclusion of the theorem. Indeed, let G be a 2-connected graph of path-width at least p. By Theorem 1.1.1, if G has tree-width at least w, then G has a minor isomorphic to $\mathcal{P}_k$, as desired. We may therefore assume that the tree-width of G is less than w. By Lemma 3.5.4 G has a minor isomorphic to $\mathcal{P}_k$ or $\mathcal{Q}_k$, as desired. $\square$

CHAPTER 4

MINORS OF 3-CONNECTED GRAPHS OF LARGE PATH-WIDTH

4.1 Properties

Let $s > 0$ be an integer. Let (T, X) be a tree decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) with size $|I| + s$ and orderings ξ_t, where I is the common intersection set of η. Let $t_0 \in V(T_h)$ be a major vertex, let (t_1, t_2, t_3) be the trinity at t_0, let G' be the η-torso at t_0, and let $i, j, k \in \{1, 2, \ldots, s\}$ be distinct.

We say that t_0 *has property* A_{ijk} *in* η if there exist disjoint paths $L_i, L_j, L_k, R_i, R_j, R_k$ in G' and vertices $y_i, y_j, y_k, z_i, z_j, z_k \in V(G')$ such that the two ends of L_m are $\xi_{t_1}(m)$ and y_m for each $m \in \{i, j, k\}$, the two ends of R_m are $\xi_{t_1}(m)$ and z_m for each $m \in \{i, j, k\}$, and $\{y_i, y_j, y_k\} = \{\xi_{t_2}(i), \xi_{t_2}(j), \xi_{t_2}(k)\}$, $\{z_i, z_j, z_k\} = \{\xi_{t_3}(i), \xi_{t_3}(j), \xi_{t_3}(k)\}$.

We say that t_0 *has property* A^0_{ijk} *in* η if there exist three disjoint tripods L_i, L_j, L_k in G' such that for each $m \in \{i, j, k\}$, the tripod L_m has feet $\xi_{t_1}(m), \xi_{t_2}(m_2), \xi_{t_3}(m_3)$ for some $m_2, m_3 \in \{i, j, k\}$. See Figure 4.1(a).

We say that t_0 *has property* A^1_{ijk} *in* η if there exist vertices $v_{x,y}$ for all $x \in \{i, j, k\}$, $y \in \{2, 3\}$, and tripods L_i, L_j, L_k in G' with centers c_i, c_j, c_k such that:

- for each $y \in \{2, 3\}$, $\{v_{i,y}, v_{j,y}, v_{k,y}\} = \{\xi_{t_y}(i), \xi_{t_y}(j), \xi_{t_y}(k)\}$
- for each $m \in \{i, j, k\}$, L_m has feet $\xi_{t_1}(m)$, $v_{m,2}, v_{m,3}$
- $L_i \cap L_j = c_i L_i v_{i,3} \cap c_j L_j v_{j,2}$ and it is a path that does not contain c_i, c_j. Let v_h be the vertex of this path that is closest to c_h for $h \in \{i, j\}$.
- $V(L_h \cap L_k) \subseteq V(c_h L_h v_h) - \{c_h, v_h\}$ for $h \in \{i, j\}$
- the paths $\xi_{t_1}(m) L_m v_{m,2}$ for all $m \in \{i, j, k\}$ are disjoint and the paths $\xi_{t_1}(m) L_m v_{m,3}$ for all $m \in \{i, j, k\}$ are disjoint.

See Figure 4.1(b).

We say that t_0 *has property* A^2_{ijk} *in* η if there exist vertices $v_{x,y}$ for all $x \in \{i, j, k\}$, $y \in \{2, 3\}$, and tripods L_i, L_j, L_k in G' with centers c_i, c_j, c_k such that:

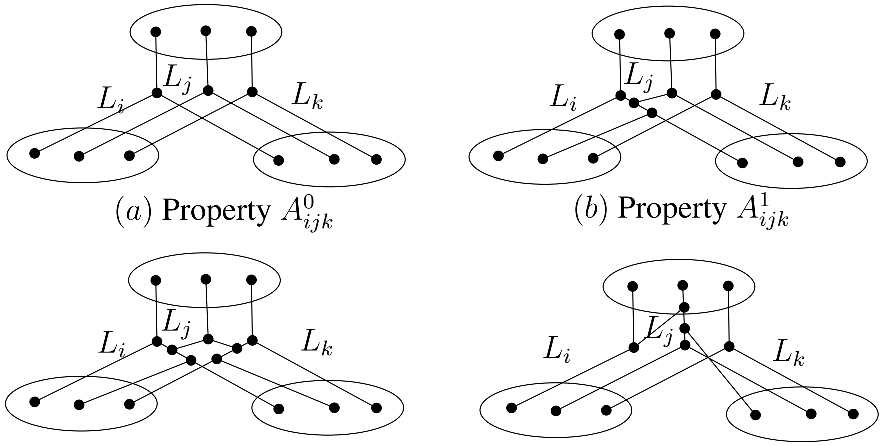
- for each $y \in \{2, 3\}$, $\{v_{i,y}, v_{j,y}, v_{k,y}\} = \{\xi_{t_y}(i), \xi_{t_y}(j), \xi_{t_y}(k)\}$
- for each $m \in \{i, j, k\}$, L_m has feet $\xi_{t_1}(m), v_{m,2}, v_{m,3}$
- $L_i \cap L_k = \emptyset$
- $L_j \cap L_i = c_j L_j v_{j,2} \cap c_i L_i v_{i,3}$ and it is a path that does not contain c_i, c_j;
  $L_j \cap L_k = c_j L_j v_{j,3} \cap c_k L_k v_{k,2}$ and it is a path that does not contain c_j, c_k.
  See Figure 4.1(c).

We say that t_0 *has property* A^3_{ijk} *in* η if there exist vertices $v_{x,y}$ for all $x \in \{i, j, k\}$ and $y \in \{2, 3\}$ such that:

- for each $y \in \{2, 3\}$, $\{v_{i,y}, v_{j,y}, v_{k,y}\} = \{\xi_{t_y}(i), \xi_{t_y}(j), \xi_{t_y}(k)\}$
- for each $m \in \{i, j, k\}$, L_m has feet $\xi_{t_1}(m), v_{m,2}, v_{m,3}$
- $L_i \cap L_k = \emptyset$ and $L_j \cap L_k = \emptyset$
- $L_i \cap L_j = c_i L_i v_{i,3} \cap c_j L_j \xi_{t_1}(j)$ and it is a path that does not contain c_i, c_j.
- there exist three disjoint paths, each from $\xi_{t_1}(h)$ to $v_{h,3}$ for $h \in \{i, j, k\}$.
  See Figure 4.1(d).

If t_0 has one of the properties above, we say that t_0 has that property *with ordered feet* if for all $h \in \{i, j, k\}$, L_h has feet $\xi_{t_1}(h), \xi_{t_2}(h), \xi_{t_3}(h)$.

If t_0 has one of the properties above, we will denote the three tripods as $L_i(t_0), L_j(t_0), L_k(t_0)$ and their centers as $c_i(t_0), c_j(t_0), c_k(t_0)$ when we want to emphasize they are in the η-torso at the major vertex t_0.



(a) Property A^0_{ijk} (b) Property A^1_{ijk} (c) Property A^2_{ijk} (d) Property A^3_{ijk}

Figure 4.1: Properties A^m_{ijk} for $m \in \{0,1,2,3\}$.

Let A_{t_0} and B_{t_0} be the confinement sets for η at t_0. We say that t_0 *has property B in* η if s is even, A_{t_0} and B_{t_0} are disjoint and both have size $s/2$, and there exist disjoint paths $R_1, R_2, \ldots, R_{3s/2}$ in G' in such a way that

- each R_i is a subpath of both the left specified t_0-linkage and the right specified t_0-linkage,
- for $i \in A_{t_0}$, the path R_i has ends $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$,
- for $i \in B_{t_0}$ the path R_i has ends $\xi_{t_1}(i)$ and $\xi_{t_3}(i)$, and
- for $i = s+1, s+2, \ldots, 3s/2$ the path R_i has one end $\xi_{t_2}(k)$ and the other end $\xi_{t_3}(l)$ for some $k \in B_{t_0}$ and $l \in A_{t_0}$.

We say that t_0 *has property* B_{ijk} *in* η if there exist three paths R_i, R_j, R_{ij} and a tripod R_k in G' such that they are pairwise disjoint and the ends of R_i are $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$, the ends of R_j are $\xi_{t_1}(j)$ and $\xi_{t_3}(j)$, the ends of R_{ij} are $\xi_{t_2}(j)$ and $\xi_{t_3}(i)$, and the three feet of R_k are $\xi_{t_1}(k)$, $\xi_{t_2}(k)$, and $\xi_{t_3}(k)$. We will denote them as $R_i(t_0), R_j(t_0), R_{ij}(t_0), R_k(t_0)$ when we want to emphasize they are in the η-torso at the major vertex t_0.

Lemma 4.1.1. *Let (T, X) be a tree-decomposition of a graph G. Let $\eta : T_1 \hookrightarrow T$ be an ordered cascade in (T, X) with orderings ξ_t of height one and size $s + |I|$, where I is the common intersection set. Let t_0 be the major vertex in T_1, and let $i, j, k \in \{1, 2, \dots, s\}$ be distinct. If t_0 has property A_{ijk} in η, then t_0 has property A^m_{abc} in η for some $m \in \{0, 1, 2, 3\}$ and a, b, c such that $\{a, b, c\} = \{i, j, k\}$.*

Proof. Assume the trinity at t_0 is (t_1, t_2, t_3). As in the definition of property A_{ijk}, in the η-torso at t_0 there exist disjoint paths L_i, L_j, L_k such that L_m has ends $\xi_{t_1}(m)$ and y_m for all $m \in \{i, j, k\}$ and there exist disjoint paths R_i, R_j, R_k such that R_m has ends $\xi_{t_1}(m)$ and z_m for all $m \in \{i, j, k\}$, $\{y_i, y_j, y_k\} = \{\xi_{t_2}(i), \xi_{t_2}(j), \xi_{t_2}(k)\}$, and $\{z_i, z_j, z_k\} = \{\xi_{t_3}(i), \xi_{t_3}(j), \xi_{t_3}(k)\}$. Let $x_m = \xi_{t_1}(m)$ for all $m \in \{i, j, k\}$. Among all the possible choices of such paths, choose the one such that $M = |\bigcup_m [E(L_m) \cup E(R_m)]|$ is minimal. Assume from z_i, z_j, z_k the paths R_i, R_j, R_k first meet $L_i \cup L_j \cup L_k$ at a, b, c, respectively.

Claim 4.1.1.1. *Let $m, n \in \{i, j, k\}$. Assume R_m meets L_n at a vertex v. Then from v, after departing from the path L_n, vR_mx_m must meet L_h before L_n for some $h \in \{i, j, k\} - \{n\}$.*

Proof: Assume it is not true. From v, assume vR_mx_m departs from L_n at a vertex v_1, and then meets L_n again before any L_h where $h \in \{i, j, k\} - \{n\}$ at a vertex v_2. Assume v_2 is closer to y_n than v_1. Let $L'_n = x_nL_nv_1 \cup v_1R_mv_2 \cup v_2L_ny_n$ and $L'_h = L_h$ for $h \in \{i, j, k\} - \{n\}$. Let $G_1 = L'_i \cup L'_j \cup L'_k \cup R_i \cup R_j \cup R_k$ and $G_2 = L_i \cup L_j \cup L_k \cup R_i \cup R_j \cup R_k$. It is clear that G_1 is a subgraph of G_2. In addition, there exists an edge of $v_1L_nv_2$ that is not an edge of $R_i \cup R_j \cup R_k$. So $|E(G_1)| < |E(G_2)| = M$, contradicting the minimality of M. □

Claim 4.1.1.2. *Let $m, n, h, l \in \{i, j, k\}$ where $m \neq n$. Let P_1 be a subpath of R_h with two ends v_1, w_1 such that $v_1 \in V(L_m), w_1 \in V(L_n)$ and P_2 be a subpath of R_l with two ends v_2, w_2 such that $v_2 \in V(L_m), w_2 \in V(L_n)$. Assume P_1, P_2 are internally disjoint from $L_i \cup L_j \cup L_k$ and P_1 is disjoint from P_2. Assume $v_1 \in V(v_2L_my_m)$. Then $w_1 \in V(w_2L_ny_n)$.*

Proof: Assume it is not true, then $w_1 \in V(w_2 L_n x_n)$. Let $L'_m = x_m L_m v_2 \cup P_2 \cup w_2 L_n y_n$ and $L'_n = x_n L_n w_1 \cup P_1 \cup v_1 L_m y_m$. Let $r \in \{i,j,k\} - \{m,n\}$. Let $G_1 = L_r \cup L'_m \cup L'_n \cup R_i \cup R_j \cup R_k$ and $G_2 = L_i \cup L_j \cup L_k \cup R_i \cup R_j \cup R_k$. Then G_1 is a subgraph of G_2. In addition, there exists an edge of $v_1 L_m v_2 \cup w_1 L_n w_2$ that is not an edge of $R_i \cup R_j \cup R_k$. So $|E(G_1)| < |E(G_2)| = M$, a contradiction. □

Claim 4.1.1.3. *Let $m,n,p \in \{i,j,k\}$ where $n \neq p$. Assume R_m meets L_n at a vertex v. From v, assume after departing from L_n, the path $vR_m x_m$ meets L_p and after departing from L_p, it meets L_n again at a vertex w. Then there exist $u \in \{a,b,c\}$ and $h \in \{i,j,k\} - \{m\}$ such that $u \in V(vL_n w) \cap V(R_h)$.*

Proof: Assume it is not true. Without loss of generality, assume $v \in V(wL_n x_n)$. Assume $vR_m w$ first meets L_p at v_1 and departs from L_p at v_2. There exist $t_1, t_2 \in V(vL_n w) \cap V(R_m)$ such that $t_1 L_n t_2$ is internally disjoint from R_m, otherwise R_m will contain a cycle. If $V(vL_n w) \cap V(\bigcup_{l \in \{i,j,k\}-\{m\}} R_l) = \emptyset$, then by changing $t_1 R_m t_2$ by $t_1 L_n t_2$, we can reduce M, a contradiction. So there exists a vertex $v_3 \in V(vL_n w) \cap V(R_{m_1})$ for some $m_1 \in \{i,j,k\} - \{m\}$. Assume from v_3 the path $v_3 R_{m_1} x_{m_1}$ departs L_n at a vertex v_4. By Claim 4.1.1.1, from v_4 the path $v_4 R_{m_1} x_{m_1}$ must meet L_{h_1} at a vertex v_5 before meeting L_n again for some $h_1 \in \{i,j,k\} - \{n\}$. Assume from v_3 the path $v_3 R_{m_1} z_{m_1}$ departs from L_n at a vertex v_6. Since $m_1 \neq m$, $v_6 \notin \{a,b,c\}$, so $v_6 R_{m_1} z_{m_1}$ must meet $L_i \cup L_j \cup L_k$. By Claim 4.1.1.1, $v_6 R_{m_1} z_{m_1}$ must meet L_{h_2} at a vertex v_7 before meeting L_n again for some $h_2 \in \{i,j,k\} - \{n\}$. By Claim 4.1.1.2, $h_1 \neq p$ and $h_2 \neq p$, so $h_1 = h_2 = r$, where $r \in \{i,j,k\} - \{n,p\}$. Assume $v_5 \in V(x_r L_r v_7)$. As above, $V(v_5 L_r v_7) \cap V(R_i \cup R_j \cup R_k) \neq \emptyset$. Let $v_8 \in V(v_5 L_r v_7) \cap V(R_{m_2})$ for some $m_2 \in \{i,j,k\}$. From v_8, assume $v_8 R_{m_2} x_{m_2}$ departs from L_r at a vertex v_9. From v_9 the path $v_9 R_{m_2} x_{m_2}$ must meet L_{h_3} at a vertex v_{10} before meeting L_r again for some $h_3 \in \{i,j,k\} - \{r\}$. By Claim 4.1.1.2, $h_3 \neq n$, so $h_3 = p$. If $v_{10} \in V(x_p L_p v_1)$, let $L'_r = x_r L_r v_5 \cup v_5 R_{m_1} v_4 \cup v_4 L_n y_n$, $L'_p = x_p L_p v_{10} \cup v_{10} R_{m_2} v_9 \cup v_9 L_r y_r$, and $L'_n = x_n L_n v \cup vR_m v_1 \cup v_1 L_p y_p$, then these paths together with R_i, R_j, R_k show that

M is not minimal. If $v_{10} \in V(v_2 L_p y_p)$, then let $L'_r = x_r L_r v_9 \cup v_9 R_{m_2} v_{10} \cup v_{10} L_p y_p$, $L'_p = x_p L_p v_2 \cup v_2 R_m w \cup w L_n y_n$, and $L'_n = x_n L_n v_6 \cup v_6 R_{m_1} v_7 \cup v_7 L_r y_r$ then we also have M is not minimal, a contradiction. $\square$

Back to the proof of the lemma, if no two of a, b, c lie on the same path in L_i, L_j, L_k, then it is clear that t_0 has property A^0_{ijk} in η. So consider two cases:

Case 1: all of a, b, c lie on some path in L_i, L_j, L_k. Without loss of generality, assume they all lie on L_j such that $b \in V(aL_jx_j)$ and $c \in V(bL_jx_j)$. From b the path bR_jx_j must depart from L_j at a vertex b_1. From b_1 the path $b_1R_jx_j$ must meet L_i or L_k before meeting L_j again. Assume from b_1 the path $b_1R_jx_j$ meets L_i at a vertex b_2. From a the path aR_ix_i must depart from L_j at a vertex a_1. From a_1 the path $a_1R_ix_i$ must meet L_i or L_k before meeting L_j again. If from a_1 the path $a_1R_ix_i$ meets L_k before L_i or L_j, then t_0 has property A^1_{kji}. So assume from a_1 the path $a_1R_ix_i$ meets L_i before L_j or L_k at a vertex a_2. By Claim 4.1.1.2, $a_2 \in V(b_2L_iy_i)$. From a_2 the path $a_2R_ix_i$ must depart from L_i at a vertex a_3. From a_3 the path $a_3R_ix_i$ must meet L_j or L_k before meeting L_i again. If from a_3 the path $a_3R_ix_i$ meets L_k before L_i or L_j, then t_0 also has property A^1_{kji}. So assume from a_3 the path $a_3R_ix_i$ meets L_j before L_i or L_k at a vertex a_4. By Claim 4.1.1.2, $a_4 \in V(bL_jy_j)$. By Claim 4.1.1.3, this cannot happen.

Case 2: exactly two of a, b, c lie on some path in L_i, L_j, L_k. Without loss of generality, assume that a and b lie on L_j and c lies on L_k and $a \in V(bL_jy_j)$. Because $b \notin V(R_i)$, from a the path aR_ix_i must depart from L_j at some vertex a_1. If $a_1R_ix_i$ meets L_i before L_j or L_k, then t_0 has property A^1_{ijk}. The path $a_1R_ix_i$ cannot meet L_j before L_i or L_k by Claim 4.1.1.1. So assume from a_1 the path $a_1R_ix_i$ meets L_k before L_i or L_j at a vertex a_2. Assume first $a_2 \in V(cL_kx_k)$. Because $a_2 \notin V(R_k)$, from c the path cR_kx_k must depart from L_k at some vertex c_1. If $c_1R_kx_k$ meets L_i before L_j or L_k, then t_0 has property A^2_{ikj}. The path $c_1R_kx_k$ cannot meet L_k before L_i or L_j because of Claim 4.1.1.1. So assume $c_1R_kx_k$ meets L_j before L_i or L_k at a vertex c_2. By Claim 4.1.1.2, $c_2 \in V(aL_jy_j)$. Because $a \notin V(c_2R_kx_k)$, from c_2 the path $c_2R_kx_k$ must depart from L_j at some vertex c_3. From

c_3 the path $c_3R_kx_k$ cannot meet L_j before L_i or L_k because of Claim 4.1.1.1. If $c_3R_kx_k$ meets L_i before L_j or L_k at a vertex c_4, then the tripods $T_1 = L_i \cup c_4R_kc_3 \cup c_3L_ja \cup aR_iz_i$, $T_2 = L_j \cup bR_jz_j$, and $T_3 = L_k \cup cR_kz_k$ show that t_0 has property A^1_{ijk}. So assume $c_3R_kx_k$ meets L_k before L_i or L_j at a vertex c_4. Because $a, b \notin V(c_1L_kc_4)$, by Claim 4.1.1.3, this cannot happen. Now assume $a_2 \in V(cL_ky_k)$. From a_2 the path $a_2R_ix_i$ must depart from L_k at a vertex a_3. From a_3 if $a_3R_ix_i$ meets L_i before L_j or L_k, then t_0 has property A^1_{kji}. From a_3 the path $a_3R_ix_i$ cannot meet L_k before L_i or L_j, so assume it meets L_j at a vertex a_4 before L_i or L_k. By Claim 4.1.1.3, $a_4 \notin V(bL_jy_j)$. So $a_4 \in V(bL_jx_j)$. Then from b the path bR_jx_j must depart from L_j at a vertex b_1. From b_1 the path $b_1R_jx_j$ cannot meet L_j again before L_i or L_k. If from b_1 the path $b_1R_jx_j$ meets L_i before L_j or L_k, then t_0 has property A^3_{ijk}. From b_1 the path $b_1R_jx_j$ cannot meet L_k before L_i or L_j because of Claim 4.1.1.2. □

4.2 Main lemma

Lemma 4.2.1. *Let $s \geq 3$ be an integer. Let (T, X) be a tree decomposition of a graph G satisfying $(W6)$, and let $\eta : T_5 \hookrightarrow T$ be a regular cascade in (T, X) of size $|I| + s$ with specified linkages that are minimal, where I is the common intersection set of η. Then there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one such that in η' the unique major vertex of T_1 has property A_{ijk} for some distinct integers $i, j, k \in \{1, 2, \dots, s\}$, or the major root of T_5 has property B in η.*

Proof. We will either construct a weakly monotone homeomorphic embedding $\gamma : T_1 \hookrightarrow T_5$ such that in $\eta' = \eta \circ \gamma$ the major root of T_1 will have property A_{ijk} for some distinct integers $i, j, k \in \{1, 2, \dots, s\}$, or establish that the major root of T_5 has property B in η.

Since η is regular, there exist sets A, B, C, D as in the definition of a regular cascade. Let t_0 be the unique major vertex of T_1 and let (t_1, t_2, t_3) be its trinity. Let u_0 be the major root of T_5 and let (v_1, v_2, v_3) to be its trinity. Let u_1, u_2 be the major vertices of T_5 of height one such that u_1 is adjacent to v_2 and u_2 is adjacent to v_3. Let (v_2, v_4, v_5) be the trinity at

u_1 and (v_3, v_6, v_7) be the trinity at u_2. Let u_3, u_4 be the major vertices of T_5 of height two such that u_3 is adjacent to v_4 and u_4 is adjacent to v_6. Let (v_4, v_8, v_9) be the trinity at u_3 and (v_6, v_{10}, v_{11}) be the trinity at u_4. Let u_5, u_6 be the major vertices of T_5 of height three such that u_5 is adjacent to v_8 and u_6 is adjacent to v_{10}. Let (v_8, v_{12}, v_{13}) be the trinity at u_5 and (v_{10}, v_{14}, v_{15}) be the trinity at u_6.

Let us recall that for a major vertex u of T_5 we denote the paths in the specified left u-linkage by $P_i(u)$ and the paths in the specified right u-linkage by $Q_i(u)$. If there exist three distinct integers $i, j, k \in A \cap B$, then the paths $P_i(u_0), P_j(u_0), P_k(u_0), Q_i(u_0), Q_j(u_0), Q_k(u_0)$ show that u_0 has property A_{ijk} in η. Let $\gamma : T_1 \hookrightarrow T_5$ be the homeomorphic embedding that maps t_0, t_1, t_2, t_3 to u_0, v_1, v_2, v_3, respectively. Then $\eta' = \eta \circ \gamma$ is as desired. We may therefore assume that $|A \cap B| \leq 2$.

For $i \in \{1, 2, \ldots, s\} - A$ the path $P_i(u_0)$ exits and re-enters the η-torso at u_0, and it does so through two distinct vertices of $X_{\eta(v_3)} - I$. But $|X_{\eta(v_3)} - I| = s$, hence $|A| \geq s/2$. Similarly $|B| \geq s/2$. Let a be a major vertex with trinity (a_1, a_2, a_3). The set C includes an element of the form (i, l, m), which means that the vertices $\xi_{a_1}(i), \xi_{a_3}(l), \xi_{a_3}(m), \xi_{a_2}(i)$ appear on the path $P_i(a)$ in the order listed. Let $l_i := l, m_i := m$, $x_i(a) := \xi_{a_3}(l)$, $y_i(a) := \xi_{a_3}(m)$, $X_i(a) := \xi_{a_1}(i)P_i(a)x_i(a)$ and $Y_i(a) := y_i(a)P_i(a)\xi_{a_2}(i)$. Thus $X_i(a)$ and $Y_i(a)$ are subpaths of the η-torso at a. Similarly, if $i \in \{1, 2, .., s\} - B$ then the set D includes an element of the form (i, n, r), which means that the vertices $\xi_{a_1}(i), \xi_{a_2}(n), \xi_{a_2}(r), \xi_{a_3}(i)$ appear on the path $Q_i(a)$ in the order listed. Let $n_i := n, r_i := r$, $w_i(a) := \xi_{a_2}(n)$, $z_i(a) := \xi_{a_2}(r), W_i(a) := \xi_{a_1}(i)Q_i(a)w_i(a)$ and $Z_i(a) := z_i(a)Q_i(a)\xi_{a_3}(i)$. We distinguish three main cases.

Main case 1: $|A \cap B| = 2$. Assume $A \cap B = \{i, j\}$. Assume $B - A = \emptyset$, then $B = \{i, j\}$. Let $k \in \{1, ..., s\} \backslash B$. Let $\gamma(t_0) = u_0, \gamma(t_1) = v_1$.

Consider the following cases depending on n_k and r_k. If $n_k, r_k \in B$ (so they are also in A), let $\gamma(t_2) = v_4$, $E_h = P_h(u_1)$ for all $h \in \{i, j, k\}$, and let L be the union of $W_k(u_0) \cup Q_{n_k}(u_1)$ and $Q_{r_k}(u_1) \cup Z_k(u_0)$ and a $W6$-path in the outer graph at v_5 joining their ends by Lemma 3.4.2. If at least one of n_k, r_k is not in B, let $\gamma(t_2) = v_5$ and $E_h = Q_h(u_1)$ for all $h \in \{i,j,k\}$. If $n_k, r_k \notin B$, let L be the union of $W_k(u_0) \cup W_{n_k}(u_1)$ and $W_{r_k}(u_1) \cup Z_k(u_0)$ and a $W6$-path in the outer graph at v_4 joining their ends by Lemma 3.4.2. If $n_k \in B$ and $r_k \notin B$, let L be the union of $W_k(u_0) \cup P_{n_k}(u_1)$ and $W_{r_k}(u_1) \cup Z_k(u_0)$ and a $W6$-path in the outer graph at v_4 joining their ends by Lemma 3.4.2. If $n_k \notin B$ and $r_k \in B$, let L be the union of $W_k(u_0) \cup W_{n_k}(u_1)$ and $P_{r_k}(u_1) \cup Z_k(u_0)$ and a $W6$-path in the outer graph at v_4 joining their ends by Lemma 3.4.2.

If $k \in A$ then we let $\gamma(t_3) = v_3$, $F_i = F_j = F_k = \emptyset$ and $R = P_k(u_0)$. If $k \notin A$ then we consider the following cases depending on l_k and m_k. If $l_k, m_k \in B$, let $\gamma(t_3) = v_6$, $F_h = P_h(u_2)$ for all $h \in \{i,j,k\}$, and let R be the union of $X_k(u_0) \cup Q_{l_k}(u_2)$ and $Q_{m_k}(u_2) \cup Y_k(u_0)$ and a $W6$-path in the outer graph at v_7 joining their ends by Lemma 3.4.2. If at least one of l_k, m_k is not in B, let $\gamma(t_3) = v_7$ and $F_h = Q_h(u_2)$ for all $h \in \{i,j,k\}$. If $l_k, m_k \notin B$, let R be the union of $X_k(u_0) \cup W_{l_k}(u_2)$ and $W_{m_k}(u_2) \cup Y_k(u_0)$ and a $W6$-path in the outer graph at v_6 joining their ends by Lemma 3.4.2. If $l_k \in B$ and $m_k \notin B$, let R be the union of $X_k(u_0) \cup P_{l_k}(u_2)$ and $W_{m_k}(u_2) \cup Y_k(u_0)$ and a $W6$-path in the outer graph at v_6 joining their ends by Lemma 3.4.2. If $l_k \notin B$ and $m_k \in B$, let R be the union of $X_k(u_0) \cup W_{l_k}(u_2)$ and $P_{m_k}(u_2) \cup Y_k(u_0)$ and a $W6$-path in the outer graph at v_6 joining their ends by Lemma 3.4.2.

Let L' be a subpath of L with the same ends and R' be a subpath of R with the same ends. Then the unique major vertex of T_1 has property A_{ijk} in $\eta' = \eta \circ \gamma$ with the first triple of disjoint paths being $R' \cup E_k$ and $P_h(u_0) \cup E_h$ for all $h \in \{i,j\}$, and the second triple being $L' \cup F_k$ and $Q_h(u_0) \cup F_h$ for all $h \in \{i,j\}$.

Now assume $B - A \neq \emptyset$. Select an element $k \in B - A$. Let $\gamma(t_0) = u_0, \gamma(t_1) = v_1, \gamma(t_2) = v_2$.

If $l_k \in A$ or $l_k \notin B$, let $\gamma(t_3) = v_7$ and $F_h = Q_h(u_2)$ for $h \in \{i,j,k\}$. If $l_k \in A$, let $M_1 = P_{l_k}(u_2)$. If $l_k \notin B$, let $M_1 = W_{l_k}(u_2)$. If $m_k \in B$, let $M_2 = Y_k(u_2) \cup Q_{m_k}(u_2)$. If $m_k \notin B$, let $M_2 = W_{m_k}(u_2)$. Let R be the union of M_1 and M_2 and a $W6$-path in the outer graph at v_6 joining their ends by Lemma 3.4.2. If $l_k \in B$ and $m_k \notin A - B$, let $\gamma(t_3) = v_6$ and $F_h = P_h(u_2)$ for all $h \in \{i, j, k\}$. Let $M_1 = Q_{l_k}(u_2)$. If $m_k \notin A$ let $M_2 = X_{m_k}(u_2)$. If $m_k \in B$ let $M_2 = Q_{m_k}(u_2)$. Let R be the union of M_1 and M_2 and a $W6$-path in the outer graph at v_7 joining their ends by Lemma 3.4.2. If $l_k \in B - A$ and $m_k \in A - B$, let $\gamma(t_3) = v_{15}$ and $F_h = P_h(u_2) \cup P_h(u_4) \cup Q_h(u_6)$ for all $h \in \{i, j, k\}$. Let $M_1 = P_{l_k}(u_2) \cup X_{l_k}(u_4)$, $M_2 = Q_j(u_4) \cup P_j(u_4) \cup P_j(u_6)$ and $M_3 = P_{m_k}(u_2) \cup P_{m_k}(u_4) \cup P_{m_k}(u_6)$. Let R be the union of M_1, M_2, M_3, a $W6$-path in the outer graph at v_{11} joining the ends of M_1 and M_2, and a $W6$-path in the outer graph at v_{14} joining the ends of M_2 and M_3 by Lemma 3.4.2.

Then the unique major vertex of T_1 has property A_{ijk} in $\eta' = \eta \circ \gamma$ with the first triple of disjoint paths being $P_h(u_0)$ for all $h \in \{i, j\}$ and a path between $\xi_{v_1}(k)$ and $\xi_{v_2}(k)$ that is a subgraph of $X_k(u_0) \cup R \cup Y_k(u_0)$, and the second triple being $Q_h(u_0) \cup F_h$ for all $h \in \{i, j, k\}$.

Main case 2: $|A \cap B| = 1$. Let j be the unique element of $A \cap B$. Notice that $A - B \neq \emptyset$. In fact, if $A - B = \emptyset$, then $|A| = 1$. So $2(s-1) \leq s$ and this means $s \leq 2$, a contradiction. Similarly, $B - A \neq \emptyset$. Therefore, we can let $i \in A - B$ and $k \in B - A$. Let $\gamma(t_0) = u_0, \gamma(t_1) = v_1$.

If $n_i \in B$ or $n_i \notin A$, let $\gamma(t_2) = v_4$ and $E_h = P_h(u_1)$ for all $h \in \{i, j, k\}$. If $n_i \in B$, let $M_1 = Q_{n_i}(u_1)$. If $n_i \notin A$, let $M_1 = X_{n_i}(u_1)$. If $r_i \in A$, let $M_2 = Z_i(u_1) \cup P_{r_i}(u_1)$. If $r_i \notin A$, let $M_2 = X_{r_i}(u_1)$. Let L be the union of M_1 and M_2 and a $W6$-path in the outer graph at v_5 joining their ends by Lemma 3.4.2. If $n_i \in A$ and $r_i \notin B - A$, let $\gamma(t_2) = v_5$ and $E_h = Q_h(u_1)$ for all $h \in \{i, j, k\}$. Let $M_1 = P_{n_i}(u_1)$. If $r_i \notin B$ let $M_2 = W_{r_i}(u_1)$. If $r_i \in A$ let $M_2 = P_{r_i}(u_1)$. Let L be the union of M_1 and M_2 and a $W6$-path in the outer graph at v_4 joining their ends by Lemma 3.4.2. If $n_i \in A - B$ and $r_i \in B - A$, let $\gamma(t_2) = v_{13}$ and $E_h = P_h(u_1) \cup P_h(u_3) \cup Q_h(u_5)$ for all $h \in \{i, j, k\}$. Let $M_1 = P_{r_i}(u_1) \cup X_{r_i}(u_3)$, $M_2 = Q_j(u_3) \cup P_j(u_3) \cup P_j(u_5)$ and $M_3 = P_{n_i}(u_1) \cup P_{n_i}(u_3) \cup P_{n_i}(u_5)$. Let L be the union of M_1, M_2, M_3, a $W6$-path in the outer graph at v_9 joining the ends of M_1 and M_2, and a $W6$-path in the outer graph at v_{12} joining the ends of M_2 and M_3 by Lemma 3.4.2.

If $l_k \in A$ or $l_k \notin B$, let $\gamma(t_3) = v_7$ and $F_h = Q_h(u_2)$ for all $h \in \{i, j, k\}$. If $l_k \in A$, let $M_1 = P_{l_k}(u_2)$. If $l_k \notin B$, let $M_1 = W_{l_k}(u_2)$. If $m_k \in B$, let $M_2 = Y_k(u_2) \cup Q_{m_k}(u_2)$. If $m_k \notin B$, let $M_2 = W_{m_k}(u_2)$. Let R be the union of M_1 and M_2 and a $W6$-path in the outer graph at v_6 joining their ends by Lemma 3.4.2. If $l_k \in B$ and $m_k \notin A - B$, let $\gamma(t_3) = v_6$ and $F_h = P_h(u_2)$ for all $h \in \{i, j, k\}$. Let $M_1 = Q_{l_k}(u_2)$. If $m_k \notin A$ let $M_2 = X_{m_k}(u_2)$. If $m_k \in B$ let $M_2 = Q_{m_k}(u_2)$. Let R be the union of M_1 and M_2 and a $W6$-path in the outer graph at v_7 joining their ends by Lemma 3.4.2. If $l_k \in B - A$ and $m_k \in A - B$, let $\gamma(t_3) = v_{15}$ and $F_h = P_h(u_2) \cup P_h(u_4) \cup Q_h(u_6)$ for all $h \in \{i, j, k\}$. Let $M_1 = P_{l_k}(u_2) \cup X_{l_k}(u_4)$, $M_2 = Q_j(u_4) \cup P_j(u_4) \cup P_j(u_6)$ and $M_3 = P_{m_k}(u_2) \cup P_{m_k}(u_4) \cup P_{m_k}(u_6)$. Let R be the union of M_1, M_2, M_3, a $W6$-path in the outer graph at v_{11} joining the ends of M_1 and M_2, and a $W6$-path in the outer graph at v_{14} joining the ends of M_2 and M_3 by Lemma 3.4.2.

Let L' be a subpath of L with the same ends and R' be a subpath of R with the same ends. Then the unique major vertex of T_1 has property A_{ijk} in $\eta' = \eta \circ \gamma$ with the first triple of disjoint paths being $P_h(u_0) \cup E_h$ for all $h \in \{i, j\}$ and $X_k(u_0) \cup R \cup Y_k(u_0) \cup E_k$, and the second triple being $Q_h(u_0) \cup F_h$ for all $h \in \{j, k\}$ and $W_i(u_0) \cup L \cup Z_i(u_0) \cup F_i$.

Main case 3: $A \cap B = \emptyset$. It follows that s is even and $|A| = |B| = s/2$. Assume as a case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$ and for some $k \in A$, $n_k, r_k \in A$ or $n_k, r_k \in B$. But the integers l_i, m_i are pairwise distinct, and so if $l_i, m_i \in A$, then there exists $j \in B$ such that $l_j, m_j \in B$, and similarly if $l_i, m_i \in B$. We may therefore assume that there exist $k \in A$ and $i, j \in B$ such that $n_k, r_k \in A$, $l_i, m_i \in A$ and $l_j, m_j \in B$. We let γ map t_0, t_1, t_2, t_3 to u_0, v_1, v_5, v_{11}, respectively, and we will prove that t_0 has property A_{ijk} in η'. To that end we need to construct two triples of disjoint paths. The first two paths of the first triple are $Q_i(u_0) \cup P_i(u_2) \cup Q_i(u_4)$ and $Q_j(u_0) \cup P_j(u_2) \cup Q_j(u_4)$. The third path of the first triple is the union of $W_k(u_0) \cup P_{n_k}(u_1)$ and $P_{r_k}(u_1) \cup Z_k(u_0) \cup P_k(u_2) \cup Q_k(u_4)$ and a suitable W6-path in the outer graph at v_4 joining their ends by Lemma 3.4.2. The first path of the second triple is $P_k(u_0) \cup Q_k(u_1)$. The second path of the second triple is the union of $X_i(u_0) \cup P_{l_i}(u_2) \cup P_{l_i}(u_4)$ and $Q_i(u_1) \cup Y_i(u_0) \cup P_{m_i}(u_2) \cup P_{m_i}(u_4)$ and a suitable W6-path in the outer graph at v_{10} joining their ends by Lemma 3.4.2. The third path of the second triple is the union of $X_j(u_0) \cup X_{l_j}(u_2)$ and $Q_j(u_1) \cup Y_j(u_0) \cup X_{m_j}(u_2)$ and a suitable W6-path in the outer a graph at v_7 joining their ends by Lemma 3.4.2. This completes the case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$ and for some integer $k \in A$ either $n_k, r_k \in A$ or $n_k, r_k \in B$.

We may therefore assume that for every $i \in B$ one of l_i, m_i belongs to A and the other belongs to B, or for every $k \in A$ one of n_k, r_k belongs to A and the other belongs to B. Without loss of generality, assume that for every $i \in B$ one of l_i, m_i belongs to A and the other belongs to B. For every $i \in B$ a subpath of $P_i(u_0)$ joins $\xi_{v_3}(l_i)$ to $\xi_{v_3}(m_i)$ in the outer graph at v_3 and is disjoint from the η-torso at u_0, except for its ends. Let J be the union of these subpaths; then J is a linkage from $\{\xi_{v_3}(i) : i \in A\}$ to $\{\xi_{v_3}(i) : i \in B\}$. For $i \in B$ the path $Q_i(u_0)$ is a subgraph of the η-torso at u_0. It follows that $J \cup (\bigcup_{i \in B} Q_i(u_0)) \cup (\bigcup_{i \in A} Z_i(u_0)) \cup (\bigcup_{i \in A} W_i(u_0))$ is a linkage from $X_{\eta(v_1)}$ to $X_{\eta(v_2)}$, and so by the minimality of the specified linkages it is equal to the specified left u_0-linkage. It follows that u_0 has property B in η. $\square$

4.3 Reduced properties

Similarly to the 2-connected case, we have the following result:

Lemma 4.3.1. *Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) with orderings ξ_t, specified linkages and common intersection set I, let $\gamma : T_{h'} \hookrightarrow T_h$ be a monotone homeomorphic embedding, and let $\eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T$ be a subcascade of η of height h'. Then for every major vertex $t_0 \in V(T_{h'})$*

(i) *η' is an ordered cascade with orderings $\xi_{\gamma(t)}$ and common intersection set I,*

(ii) *if the vertex $\gamma(t_0)$ has property A^m_{ijk} (or B_{ijk}, resp.) in η, then t_0 has property A^m_{ijk} (or B_{ijk}, resp.) in η'.*

Furthermore, the specified linkages for η' may be chosen in such a way that

(iii) $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0}) = (A_{\gamma(t_0)}, B_{\gamma(t_0)}, C_{\gamma(t_0)}, D_{\gamma(t_0)})$,

(iv) *the vertex t_0 has property B in η' if and only if $\gamma(t_0)$ has property B in η, and*

(v) *if the specified linkages for η are minimal, then the specified linkages for η' are minimal.*

Lemma 4.3.2. *There exists a positive integer h such that the following holds. Let $s \geq 3$ be an integer and let (T, X) be a tree-decomposition of a graph G. Let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) of height h and size $|I| + s$ with orderings ξ_t and common intersection set I such that there exist some distinct $i, j, k \in \{1, 2, \ldots, s\}$ and $m \in \{0, 1, 2, 3\}$ such that every major vertex of T_h has property A^m_{ijk}. Then there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one such that the unique major vertex of T_1 has property A^m_{ijk} with ordered feet in η'.*

Proof. Let $h(a, k)$ be the function of Lemma 3.3.2. Let $h = h(3, (3!)^2)$. Assume u is an arbitrary major vertex of T_h and its trinity is (v_1, v_2, v_3). Assume the feet of L_i, L_j, L_k in $X_{\eta(v_2)}$ are x_1, x_2, x_3 and the feet of L_i, L_j, L_k in $X_{\eta(v_3)}$ are x_4, x_5, x_6. Then for every major vertex u of T_h, consider the tuple $(x_1, x_2, x_3, x_4, x_5, x_6)$ as its color. By Lemma 3.3.2, there exists a monotone homeomorphic embedding $\gamma : T_3 \hookrightarrow T_h$ such that $\gamma(t)$ has the same tuple of six feet for every major vertex $t \in V(T_3)$. Let $\eta_1 = \eta \circ \gamma : T_3 \hookrightarrow T$. By Lemma 4.3.1, η_1 is still an ordered cascade where every major vertex $t \in V(T_3)$ has property A^m_{ijk}. Also, t has the same tuple of six feet for every major vertex $t \in V(T_3)$.

Let u be a major vertex in T_3 and let (v_1, v_2, v_3) be its trinity. Let x_i, x_j, x_k be the feet of $L_i(u), L_j(u), L_k(u)$ in $X_{\eta_1(v_1)}$, respectively. Let f, g be functions such that $f(x_l)$ are the feet of $L_l(u)$ in $X_{\eta_1(v_2)}$ and $g(x_l)$ are the feet of $L_l(u)$ in $X_{\eta_1(v_3)}$ for all $l \in \{i, j, k\}$. Define $f_0(x) = f(x)$ and $f_n(x) = f(f_{n-1}(x))$ for $n \geq 1$, and $g_0(x) = g(x)$ and $g_n(x) = g(g_{n-1}(x))$ for $n \geq 1$.

Assume t_0 is a major root of T_1 and its trinity is (t_1, t_2, t_3). Let u_0 be the major root of T_3 and its trinity be (v, v_1, w_1). Let $\gamma_1(t_0) = u_0$ and $\gamma_1(t_1) = v$. For $l \in \{1, 2\}$, let u_i be the child of v_i and v_{i+1} be the left child of u_i, and let r_i be the child of w_i and w_{i+1} be the right child of r_i. Let x_i, x_j, x_k be the feet of $L_i(u_0), L_j(u_0), L_k(u_0)$ in $X_{\eta(v)}$. Then there exist $l_1, l_2 \in \{1,2,3\}$ such that $f_{l_1}(x) = x$ and $g_{l_2}(x) = x$ for all $x \in \{x_i, x_j, x_k\}$. Let $\gamma_1(t_2) = v_{l_1}$,$\gamma_1(t_3) = w_{l_2}$, and $\eta' = \eta_1 \circ \gamma_1$. For $l \in \{i,j,k\}$, let

$$L_l = L_l(u_0) \cup \big(\bigcup_{1 \leq n < l_1} f_n(x_l) L_l(u_n) f_{n+1}(x_l) \big) \cup \big(\bigcup_{1 \leq n < l_2} g_n(x_l) P_l(r_n) g_{n+1}(x_l) \big),$$

where $g_n(x_l)P_l(r_n)g_{n+1}(x_l) = g_n(x_l)L_l(r_n)g_{n+1}(x_l)$ when $m \neq 3$, otherwise $g_n(x_l)P_l(r_n)g_{n+1}(x_l)$ for all $l \in \{i,j,k\}$ are three disjoint paths as in the definition of property A^3_{ijk}. Then these tripods show that η' is as desired. $\square$

Lemma 4.3.3. *Let $s \geq 3$ be an integer and let (T,X) be a tree-decomposition of a graph G satisfying (W6). Let $\eta : T_2 \hookrightarrow T$ be an ordered cascade in (T,X) of height two and size $|I| + s$ with orderings ξ_t and common intersection set I such that there exist distinct $i,j,k \in \{1,2,\ldots,s\}$ such that every major vertex of T_2 has property A^2_{ijk} with ordered feet. Then there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one such that the unique major vertex of T_1 has property A^1_{ikj} in η' with ordered feet.*

Proof. Assume that the major root of T_2 is u_0 and its trinity is (v_1, v_2, v_3). Let u_1 be the major vertex at height one that is adjacent to v_2 and let v_4 be its left child. Let the major root of T_1 be t_0 and its trinity be (t_1,t_2,t_3). Let $\gamma(t_0) = u_0$, $\gamma(t_1) = v_1$, $\gamma(t_2) = v_4$, $\gamma(t_3) = v_3$. Then $\eta' = \eta \circ \gamma$ is as desired. Let a be the end of $L_j(u_0) \cap L_k(u_0)$ that is closest to $\xi_{v_3}(j)$, b be the end of $L_j(u_1) \cap L_k(u_1)$ that is closest to $c_j(u_1)$, c be the end of $L_i(u_1) \cap L_j(u_1)$ that is closest to $c_i(u_1)$, and d be the end of $L_i(u_0) \cap L_j(u_0)$ that is closest to $\xi_{v_3}(i)$. Let

$$L_i = \xi_{v_1}(i)L_i(u_0)\xi_{v_2}(i) \cup \xi_{v_2}(i)L_i(u_1)\xi_{v_4}(i) \cup \xi_{v_3}(j)L_j(u_0)a \cup aL_k(u_0)\xi_{v_2}(k) \cup$$

$$\cup\, \xi_{v_2}(k)L_k(u_1)b \cup bL_j(u_1)c_j(u_1) \cup c_j(u_1)L_j(u_1)c \cup cL_i(u_1)c_i(u_1),$$

$$L_j = \xi_{v_1}(j)L_j(u_0)\xi_{v_2}(j) \cup \xi_{v_2}(j)L_j(u_1)\xi_{v_4}(j) \cup \xi_{v_3}(i)L_i(u_0)d,$$

and

$$L_k = L_k(u_0) \cup \xi_{v_2}(k) L_k(u_1) \xi_{v_4}(k),$$

then they are the tripods needed for property A^1_{ikj} in η'. $\square$

Lemma 4.3.4. *Let $s \geq 3$ be an integer and let (T, X) be a tree-decomposition of a graph G. Let $\eta : T_2 \hookrightarrow T$ be an ordered cascade in (T, X) of height two and size $|I| + s$ with orderings ξ_t and common intersection set I such that there exist distinct $i, j, k \in \{1, 2, \dots, s\}$ such that every major vertex of T_2 has property A^3_{ijk} with ordered feet. Then there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one such that the unique major vertex of T_1 has property A^1_{ijk} in η'.*

Proof. Assume that the major root of T_2 is u_0 and its trinity is (v_1, v_2, v_3). Let u_1 be the major vertex at height one that is adjacent to v_2 and let v_4 be its left child. Let the major root of T_1 be t_0 and its trinity be (t_1, t_2, t_3). Let $\gamma(t_0) = u_0$, $\gamma(t_1) = v_1$, $\gamma(t_2) = v_4$, $\gamma(t_3) = v_3$. Then $\eta' = \eta \circ \gamma$ is as desired. Assume a is the end of $L_i(u_0) \cap L_j(u_0)$ that is closest to $\xi_{v_3}(i)$, and b is the end of $L_i(u_1) \cap L_j(u_1)$ that is closest to $c_i(u_1)$. Let

$$L_i = \xi_{v_1}(i) L_i(u_0) \xi_{v_2}(i) \cup \xi_{v_2}(i) L_i(u_1) \xi_{v_4}(i) \cup \xi_{v_3}(j) L_j(u_0) c_j(u_0) \cup c_j(u_0) L_j(u_0) \xi_{v_2}(j) \cup$$

$$\cup\, \xi_{v_2}(j) L_j(u_1) b \cup b L_i(u_1) c_i(u_1),$$

$$L_j = \xi_{v_1}(j) L_j(u_0) \xi_{v_2}(j) \cup \xi_{v_2}(j) L_j(u_1) \xi_{v_4}(j) \cup \xi_{v_3}(i) L_i(u_0) a,$$

and

$$L_k = L_k(u_0) \cup \xi_{v_2}(k) L_k(u_1) \xi_{v_4}(k),$$

then they are the tripods needed for property A^1_{ijk} in η'. $\square$

Lemma 4.3.5. *For every integer $s \geq 3$ there exists a positive integer h such that the following holds. Let (T, X) be a tree-decomposition of a graph G satisfying (W6) and (W7). Let $\eta : T_h \hookrightarrow T$ be a regular cascade in (T, X) of height h and size $|I| + s$ with* *orderings ξ_t and common intersection set I such that every major vertex of T_h has property B. Then there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one and distinct $i, j, k \in \{1, 2, ..., s\}$ such that the unique major vertex of T_1 has property B_{ijk} in η'.*

Proof. Let h be as in Lemma 3.3.2 applied to $a = 3$ and $k = (s/2)^2 + 2(s/2)^3$. Let the common confinement sets for η be A, B, C, D. Let the major root of T_1 be t_0 and its trinity be (t_1, t_2, t_3). Let the major root of T_2 be u_0 and its trinity be (w_1, w_2, w_3). Let two major vertices at height one of T_2 be u_1 and u_2. Assume the trinity at u_1 is (w_2, w_4, w_5) and the trinity at u_2 is (w_3, w_6, w_7).

For a major vertex $w \in V(T_h)$ with trinity (v_1, v_2, v_3) there are disjoint paths in the η-torso at w as in the definition of property B. For $a \in A$ and $b \in B$ let $R_a(w)$ denote the path with ends $\xi_{v_1}(a)$ and $\xi_{v_2}(a)$, let $R_b(w)$ denote the path with ends $\xi_{v_1}(b)$ and $\xi_{v_3}(b)$, and let $R_{ab}(w)$ denote the path with ends $\xi_{v_2}(b)$ and $\xi_{v_3}(a)$.

Let I be the common intersection set of η. Then $\eta(v_1), \eta(v_2), \eta(v_3)$ is a triad in T with center $\eta(w)$ and for all $i \in \{1, 2, 3\}$ we have $X_{\eta(v_i)} \cap X_{\eta(w)} = I = X_{\eta(v_1)} \cap X_{\eta(v_2)} \cap X_{\eta(v_3)}$, and hence the triad is not X-separable. By (W7) there is a path $R(w)$ connecting two of the three sets of disjoint paths in the η-torso at w.

If $R(w)$ goes from $R_a(w)$ to $R_b(w)$ for $a \in A$ and $b \in B$, we say it w has color (a, b). If $R(w)$ goes from $R_a(w)$ to R_{cb} for $a \in A$ or $a \in B$ and $b \in B, c \in A$, we say w has color (a, cb). By Lemma 3.3.2, there exists a monotone homeomorphic embedding $\gamma : T_3 \hookrightarrow T_h$ and $a \in A, b \in B$ such that $\gamma(t)$ has color (a, b) in η for every major vertex $t \in V(T_3)$, or there exists a monotone homeomorphic embedding $\gamma : T_3 \hookrightarrow T_h$ and $a \in A$ or $a \in B$ and $b \in B, c \in A$ such that $\gamma(t)$ has color (a, cb) in η for every major vertex $t \in V(T_3)$.

Assume there exists a monotone homeomorphic embedding $\gamma : T_3 \hookrightarrow T_h$ and $a \in A, b \in B$ such that $\gamma(t)$ has color (a, b) in η for every major vertex $t \in V(T_3)$. Let $\eta_1 = \eta \circ \gamma$, then by Lemma 4.3.1, t has property B in η_1 for every major vertex $t \in V(T_3)$ and one end of $R(t)$ is in the path $R_a(t)$ and the other end is in $R_b(t)$. Let $\gamma_1(t_0) = u_0, \gamma_1(t_1) = w_1, \gamma_1(t_2) = w_4$, and $\gamma_1(t_3) = w_6$. Let $\eta' = \eta_1 \circ \gamma_1$. Let $c \in A - \{a\}$ and

$d \in B - \{b\}$. Let $x_1 \in B$ be such that $R_{cx_1}(u_0)$ is a member of the family of the disjoint paths in the η-torso at u_0 as in the definition of property B, $x_2 \in A$ be such that $R_{x_2d}(u_1)$ is a member of the family of the disjoint paths in the η-torso at u_1 as in the definition of property B, and $x_3 \in A$ be such that $R_{x_3d}(u_2)$ is a member of the family of the disjoint paths in the η-torso at u_2 as in the definition of property B. Let y be the end of $R(u_0)$ in the path $R_b(u_0)$, z be the end of $R(u_2)$ in the path $R_b(u_2)$, and r be the end of $R(u_2)$ in the path $R_a(u_2)$. Let $R_c = R_c(u_0) \cup R_c(u_1)$, R_d be the union of $R_d(u_0) \cup R_d(u_2)$ and $R_{x_3d}(u_2)$ and a W6-path in the outer graph at w_7 joining their ends, R_{cd} be the union of $R_{x_2d}(u_1)$ and $R_{x_1}(u_1) \cup R_{cx_1}(u_0) \cup R_c(u_2)$ and a W6-path in the outer graph at w_5 connecting the ends of these two paths, and $R_a = R_a(u_0) \cup R_a(u_1) \cup R(u_0) \cup yR_b(u_0)\xi_{w_3}(b) \cup \xi_{w_3}(b)R_b(u_1)z \cup R(u_2) \cup rR_a(u_2)\xi_{w_6}(a)$, then these paths and tripod show that t_0 has property B_{cda} in η', so η' is as desired. See Figure 4.2.

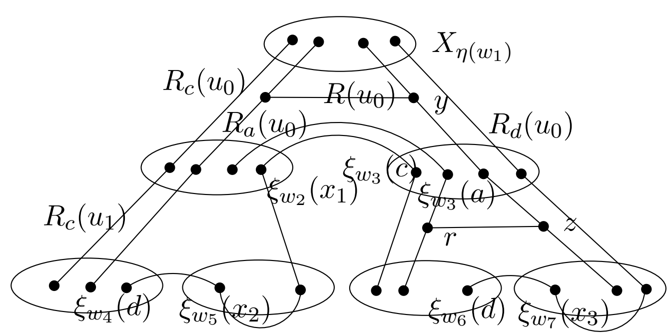


Figure 4.2: $\gamma(t)$ has color (a, b) in η for every $t \in V(T_3)$.

Therefore we can assume there exists a monotone homeomorphic embedding $\gamma : T_3 \hookrightarrow T_h$ and $a \in A$ or $a \in B$ and $b \in B, c \in A$ such that $\gamma(t)$ has color (a, cb) in η for every major vertex $t \in V(T_3)$. Without loss of generality, assume $a \in A$. Let $\eta_1 = \eta \circ \gamma$, then by Lemma 4.3.1, t has property B in η_1 for every major vertex $t \in V(T_3)$ and one end of $R(t)$ is in the path $R_a(t)$ and the other end is in $R_{cb}(t)$. Let $d \in B - \{b\}$. If $a = c$ then let $e \in A - \{a\}$ such that $R_{ed}(u_0)$ is a member of the family of the disjoint paths in the η-torso at u_0 as in the definition of property B. Let $\gamma_1(t_0) = u_0$ and $\gamma_1(t_l) = w_l$ for all $l \in \{1, 2, 3\}$, then t_0 has property B_{eda} in $\eta' = \eta_1 \circ \gamma_1$, so η' is as desired. Therefore assume $a \neq c$. Let $x_1 \in B$ be such that $R_{ax_1}(u_2)$ is a member of the family of the disjoint paths in the η-torso at u_2 as in the definition of property B. Let $f \in B$ be such that $R_{af}(u_0)$ is a member of the family of the disjoint paths in the η-torso at u_0 as in the definition of property B. Then $x_1 \neq b$ and $f \neq b$ because $a \neq c$. Let y be the end of $R(u_0)$ in the path $R_{cb}(u_0)$, z be the end of $R(u_2)$ in the path $R_a(u_2)$, and r be the end of $R(u_2)$ in the path $R_{cb}(u_2)$. Let $\gamma_1(t_0) = u_0$, $\gamma_1(t_l) = w_l$ for all $l \in \{1, 2\}$, and $\gamma_1(t_3) = w_7$. Let $\eta' = \eta_1 \circ \gamma_1$. Let $R_c = R_c(u_0)$, $R_f = R_f(u_0) \cup R_f(u_2)$, $R_{cf} = R_{af}(u_0) \cup \xi_{w_3}(a)R_a(u_2)z \cup R(u_2) \cup rR_{cb}(u_2)\xi_{w_7}(c)$, and R_a be the union of $R_a(u_0) \cup R(u_0) \cup yR_{cb}(u_0)\xi_{w_3}(c) \cup R_c(u_2)$ and $R_{ax_1}(u_2)$ and a W6-path in the outer graph at w_6 connecting the ends of these two paths, then they show that t_0 has property B_{cfa} in η'. See Figure 4.3. Hence η' is as desired. $\square$

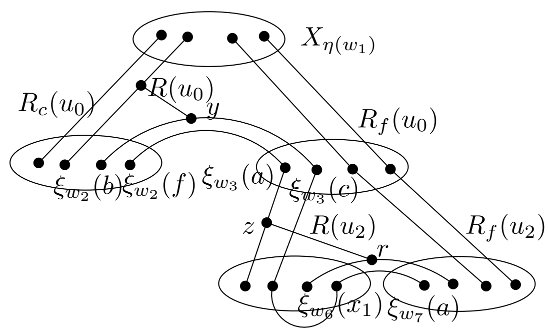


Figure 4.3: $\gamma(t)$ has color (a, cb) in η for every $t \in V(T_3)$.

Lemma 4.3.6. *For every positive integers h' and $w \geq 3$ there exists a positive integer $h = h(h', w)$ such that the following holds. Let s be a positive integer such that $3 \leq s \leq w$. Let (T, X) be a tree-decomposition of a graph G of width less than w and satisfying (W6)-(W7). Assume there exists a regular cascade $\eta : T_h \hookrightarrow T$ of size $|I| + s$ with specified linkages that are minimal, where I is its common intersection set. Then there exist distinct integers $i, j, k \in \{1, 2, \ldots, s\}$ and a weak subcascade $\eta' : T_{h'} \hookrightarrow T$ of η of height h' such that*

- *every major vertex of $T_{h'}$ has property A^0_{ijk} with ordered feet in η', or*

- *every major vertex of $T_{h'}$ has property A^1_{ijk} with ordered feet in η'*

- *every major vertex of $T_{h'}$ has property B_{ijk} in η'*

Proof. Let $h(a,k)$ be the function of Lemma 3.3.2. Let h_1 be h in Lemma 4.3.2 and h_2 be h in Lemma 4.3.5. Let $a_4 = 2h'$, $a_3 = h_1 a_4$, $a_2 = h(a_3, 12\binom{w}{3})$, $a_1 = \max\{5a_2, h_2 a_2\}$ and $h = h(a_1, 2)$. Consider having property B or not having property B as colors, then by Lemma 3.3.2 there exists a monotone homeomorphic embedding $\gamma : T_{a_1} \hookrightarrow T_h$ such that either $\gamma(t)$ has property B in η for every major vertex $t \in V(T_{a_1})$ or $\gamma(t)$ does not have property B in η for every major vertex $t \in V(T_{a_1})$. By Lemma 4.3.1 $\eta_1 = \eta \circ \gamma : T_{a_1} \hookrightarrow T$ is still a regular cascade with specified linkages that are minimal. Also, either t has property B in η_1 for every major vertex $t \in V(T_{a_1})$ or t does not have property B in η_1 for every major vertex $t \in V(T_{a_1})$.

If t has property B in η_1 for every major vertex $t \in V(T_{a_1})$, then by Lemma 4.3.5 there exists a weak subcascade η_2 of η_1 of height a_2 such that every major vertex of T_{a_2} has property B_{ijk} in η_2 for some distinct $i,j,k \in \{1,2,...,s\}$. Consider each choice of tuple (i,j,k) as a color; then by Lemma 3.3.2 there exists a monotone homeomorphic embedding $\gamma_1 : T_{a_3} \hookrightarrow T_{a_2}$ such that for some distinct $i,j,k \in \{1,2,...,s\}$, $\gamma_1(t)$ has property B_{ijk} in η_2 for every major vertex $t \in V(T_{a_3})$. Let $\eta_3 = \eta_2 \circ \gamma_1$. Then by Lemma 4.3.1 this implies t has property B_{ijk} in η_3 for every major vertex $t \in V(T_{a_3})$. Hence η_3 is as desired.

If t does not have property B in η_1 for every major vertex $t \in V(T_{a_1})$, then by Lemma 4.2.1 there exists a weak subcascade η_2 of η_1 of height a_2 such that every major vertex of T_{a_2} has property A_{ijk} for some distinct $i,j,k \in \{1,2,...,s\}$. By Lemma 4.1.1, every major vertex of T_{a_2} has property A^m_{ijk} for some distinct $i,j,k \in \{1,2,...,s\}$ and $m \in \{0,1,2,3\}$. Consider each property A^m_{ijk} as a color; then by Lemma 3.3.2 there exists a monotone homeomorphic embedding $\gamma_1 : T_{a_3} \hookrightarrow T_{a_2}$ such that for some distinct $i,j,k \in \{1,2,...,s\}$ and $m \in \{0,1,2,3\}$, $\gamma_1(t)$ has property A^m_{ijk} in η_2 for every major vertex $t \in V(T_{a_3})$.

Let $\eta_3 = \eta_2 \circ \gamma_1$, then t has property A^m_{ijk} in η_3 for every major vertex $t \in V(T_{a_3})$ by Lemma 4.3.1. By Lemma 4.3.2, there exists a weak subcascade η_4 of η_3 of height a_4 such that every major vertex of T_{a_4} has property A^m_{ijk} with ordered feet. If $m \in \{0, 1\}$, then η_4 is as desired. If $m = 2$ (or $m = 3$, resp.), then by Lemma 4.3.3 (or Lemma 4.3.4, resp.), there exists a weak subcascade η_5 of η_4 of height h' such that every major vertex of $T_{h'}$ has property A^m_{ijk} with ordered feet. Then η_5 is as desired. □

4.4 Proof of Theorem 1.1.5

Lemma 4.4.1. *If a graph H has two distinct vertices u, v such that $H\backslash\{u, v\}$ is a forest, then there exists an integer n such that H is isomorphic to a minor of $\mathcal{P}'_n$.*

Proof. Let u and v be such that $T := H\backslash\{u, v\}$ is a forest. We may assume, by replacing H by a graph with an H minor, that T is isomorphic to CT_t for some t, and that each of u, v is adjacent to every vertex of T. It follows that H is isomorphic to a minor of $\mathcal{P}'_{2t}$, as desired. □

Lemma 4.4.2. *Let H be a graph with a vertex v such that $H\backslash\{v\}$ is an outerplanar graph. Then there exists an integer n such that H is isomorphic to a minor of $\mathcal{Q}'_n$.*

Proof. By Lemma 3.1.4, there exists an integer t such that $H\backslash\{v\}$ is isomorphic to a minor of $\mathcal{Q}_t$. We may assume, by replacing H by a graph with an H minor, that $H\backslash\{v\}$ is isomorphic to $\mathcal{Q}_t$ for some t, and that v is adjacent to every vertex of $\mathcal{Q}_t$. It follows that H is isomorphic to a minor of $\mathcal{Q}'_{t+2}$. □

Lemma 4.4.3. *Let H be a tree with a cycle going through its leaves in order from the leftmost leaf to the rightmost leaf. Then there exists an integer n such that H is isomorphic to a minor of $\mathcal{R}'_n$.*

Proof. Let T be the tree in H and C be the cycle going through its leaves. We may assume, by replacing H by a graph with an H minor, that T is isomorphic to CT_t for some t, and that C goes through its leaves in order from the leftmost leaf to the rightmost leaf. It follows that H is isomorphic to a minor of $\mathcal{R}'_t$, as desired. $\square$

By Lemmas 4.4.1, 4.4.2 and 4.4.3 Theorem 1.1.5 is equivalent to the following theorem.

Theorem 4.4.4. *For every positive integer n, there exists an integer $p = p(n)$ such that every 3-connected graph with path-width at least p has $\mathcal{P}'_n$, $\mathcal{Q}'_n$ or $\mathcal{R}'_n$ as a minor.*

Lemma 4.4.5. *For every two positive integers n and $w \geq 3$ there exists an integer h such that the following holds. Let (T, X) be a tree-decomposition of a graph G of width less than w and satisfying (W1)–(W7). Assume there exists a regular cascade $\eta : T_h \hookrightarrow T$ of size $|I| + s$ with specified linkages that are minimal, where I is its common intersection set and $3 \leq s \leq w$. Then*

(i) G has a minor isomorphic to $\mathcal{P}'_n$ or $\mathcal{R}'_n$.

(ii) If $|I| \geq 1$, then G has a minor isomorphic to $\mathcal{P}''_n$ or $\mathcal{R}''_n$.

Proof. Let $h' = 4n + 1$, and let $h = h(h', w)$ be the number as in Lemma 4.3.6. By Lemma 4.3.6 there exist distinct integers $i, j, k \in \{1, 2, \ldots, s\}$ and a weak subcascade $\eta' : T_{h'} \hookrightarrow T$ of η of height h' such that

- every major vertex of $T_{h'}$ has property A^0_{ijk} with ordered feet in η', or
- every major vertex of $T_{h'}$ has property A^1_{ijk} with ordered feet in η'
- every major vertex of $T_{h'}$ has property B_{ijk} in η'

Assume that every major vertex of $T_{h'}$ has property A^0_{ijk} with ordered feet in η', and let R be the union of the corresponding tripods, over all major vertices $t \in V(T_{h'})$ at height at most $h' - 2$. It follows that R is the union of three disjoint trees, each containing a subtree isomorphic to $T_{(h'-1)/2}$. Let t be a minor vertex of $T_{h'}$ at height $h' - 1$. By Lemma 3.4.2 there exist a W6-path with ends $\xi_t(i)$ and $\xi_t(k)$ and a W6-path with ends $\xi_t(j)$ and $\xi_t(k)$ in the outer graph at t. Let R_1 be the union of these W6-paths for all minor vertices t at height $h' - 1$. By contracting the tree that contains $\xi_t(i)$ and the tree that contains $\xi_t(j)$, and by considering the remaining tree and R_1 we deduce that G has a $\mathcal{P}'_n$ minor, as desired. If $|I| \geq 1$, assume $x \in I$. By Lemma 3.4.2 there exists a W6-path with ends x and $\xi_t(k)$ in the outer graph at t. Let R_2 be the union of these W6-paths for all minor vertices t at height $h' - 1$. By contracting the tree that contains $\xi_t(i)$ and the tree that contains $\xi_t(j)$, and by considering the remaining tree and R_1, R_2 we deduce that G has a $\mathcal{P}''_n$ minor, as desired.

Assume next that every major vertex of $T_{h'}$ has property A^1_{ijk} with ordered feet in η'. Let the major root of $T_{h'}$ be u_0 and its left child be v. For every major vertex u that is a descendant of v, let $L_i(u), L_j(u), L_k(u)$ be the three tripods in the η'-torso at u as in the definition of property A^1_{ijk}, and let $a(u), b(u)$ be the two ends of the path $L_i(u) \cap L_j(u)$. Let

$$R_1 = \bigcup_u (\xi_{v_1}(i) L_i(u) \xi_{v_2}(i) \cup \xi_{v_1}(j) L_j(u) \xi_{v_3}(j) \cup$$

$$\cup\, \xi_{v_2}(j) L_j(u) a(u) \cup a(u) L_j(u) b(u) \cup b(u) L_i(u) \xi_{v_3}(i)),$$

and

$$R_2 = \bigcup_u L_k(u),$$

where the unions are taken over all major vertices u at height at most $h' - 2$ that are descendants of v and (v_1, v_2, v_3) here is the trinity at u. Then R_1 is disjoint from R_2, which is a tree isomorphic to a subdivision of $T_{h'-2}$. Let t be a minor vertex of $T_{h'}$ at height $h' - 1$. By Lemma 3.4.2 there exist a W6-path with ends $\xi_t(i)$ and $\xi_t(k)$ and a W6-path with ends $\xi_t(j)$ and $\xi_t(k)$ in the outer graph at t. Let R_3 be the union of these W6-paths for all minor vertices t at height $h' - 1$. By Lemma 3.4.2, there exists a W6-path P with ends $\xi_v(i)$ and $\xi_v(j)$ in the subgraph of G induced by $\bigcup X_r - I$, where the union is taken over all r in the component containing $\eta'(u_0)$ of $T - \eta'(v)$. By considering R_1, R_2, R_3 and P we deduce that G has a $\mathcal{R}'_{h'-3}$ minor, as desired. If $|I| \geq 1$, assume $x \in I$. By Lemma 3.4.2 there exists a W6-path with ends x and $\xi_t(k)$ in the outer graph at t. Let R_4 be the union of these W6-paths for all minor vertices t at height $h' - 1$. By considering R_1, R_2, R_3, P and R_4 we deduce that G has a $\mathcal{R}''_n$ minor, as desired.

We may therefore assume that every major vertex of $T_{h'}$ has property B_{ijk} in η'. For every major vertex u in $T_{h'}$, let $R_i(u), R_j(u), R_{ij}(u)$ and $R_k(u)$ be as in the definition of property B_{ijk}. Let the major root of $T_{h'}$ be u_0 and its left child be v. Let

$$R_1 = \bigcup_u \big(R_i(u) \cup R_j(u) \cup R_{ij}(u)\big) \text{ and } R_2 = \bigcup_u R_k(u),$$

where the unions are taken over all major vertices u at height at most $h' - 2$ that are descendants of v. Then R_1 is disjoint from R_2, which is a tree isomorphic to a subdivision of $T_{h'-2}$. By considering R_1, R_2 and the W6-paths as in the above case, we deduce that G also has a $\mathcal{R}'_{h'-3}$ minor and a $\mathcal{R}''_n$ minor if $|I| \geq 1$, as desired. □

Lemma 4.4.6. *Let n and w be positive integers. There exists an integer $p = p(n, w)$ such that for every 3-connected graph G, if G has tree-width less than w and path-width at least p, then G has a minor isomorphic to $\mathcal{P}'_n$, $\mathcal{Q}'_n$ or $\mathcal{R}'_n$.*

Proof. Let h_1 be as in Lemma 3.5.3 applied to $k = n$ and w. Let h_2 be as in Lemma 4.4.5. Let $h = \max\{h_1, h_2, n + 2\}$. Let p be as in Theorem 3.3.5 applied to $a = h$ and w. By Theorem 3.3.5, there exists a tree-decomposition (T, X) of G such that:

- (T, X) has width less than w,
- (T, X) satisfies (W1)–(W7), and
- for some s, where $3 \leq s \leq w$, there exists a regular cascade $\eta : T_h \hookrightarrow T$ of height h and size s in (T, X) with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_h)$.

Let I be the common intersection set of η, let ξ_t be the orderings, and let $s_1 = s - |I|$. Then $s_1 \geq 1$ by the definition of injective cascade.

Assume that $s_1 = 1$. Since $s \geq 3$, it follows that $|I| \geq 2$. Let $x, y \in I$. Let R be the union of the left and right specified t-linkage with respect to η, over all major vertices

$t \in V(T_h)$ at height at most $h-2$. The minimality of the specified linkages implies that R is isomorphic to a subdivision of T_{h-1}. Let t be a minor vertex of T_h at height $h-1$. By Lemma 3.4.2 there exist a W6-path with ends $\xi_t(1)$ and x and a W6-path with ends $\xi_t(1)$ and y in the outer graph at t. The union of R and these W6-paths shows that G has a $\mathcal{P}'_n$ minor, as desired.

Assume that $s_1 = 2$. Since $s \geq 3$, it follows that $I \neq \emptyset$. By Lemma 3.5.3(ii), G has a $\mathcal{P}'_n$ minor or a $\mathcal{Q}'_n$ minor, as desired.

We may therefore assume that $s_1 \geq 3$. By Lemma 4.4.5(i), G has a minor isomorphic to $\mathcal{P}'_n$ or $\mathcal{R}'_n$, as desired. $\square$

Proof of Theorem 4.4.4. Let a positive integer n be given. By Theorem 1.1.1 there exists an integer w such that every graph of tree-width at least w has a minor isomorphic to $\mathcal{Q}'_n$. Let $p = p(n, w)$ be as in Lemma 4.4.6. We claim that p satisfies the conclusion of the theorem. Indeed, let G be a 3-connected graph of path-width at least p. By Theorem 1.1.1, if G has tree-width at least w, then G has a minor isomorphic to $\mathcal{Q}'_n$, as desired. We may therefore assume that the tree-width of G is less than w. By Lemma 4.4.6 G has a minor isomorphic to $\mathcal{P}'_n$, $\mathcal{Q}'_n$ or $\mathcal{R}'_n$, as desired. $\square$

CHAPTER 5

MINORS OF 4-CONNECTED GRAPHS OF LARGE PATH-WIDTH

5.1 Properties

Let $s > 0$ be an integer. Let (T, X) be a tree decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) with size $|I| + s$ and orderings ξ_t, where I is the common intersection set of η. Let $t_0 \in V(T_h)$ be a major vertex, let (t_1, t_2, t_3) be the trinity at t_0, let G' be the η-torso at t_0, and let $i, j, k, l \in \{1, 2, \ldots, s\}$ be distinct.

We say that t_0 *has property* A_{ijkl} *in* η if there exist disjoint paths $L_i, L_j, L_k, L_l, R_i, R_j, R_k, R_l$ in G' and vertices $y_i, y_j, y_k, y_l, z_i, z_j, z_k, z_l \in V(G')$ such that the two ends of L_m are $\xi_{t_1}(m)$ and y_m for each $m \in \{i, j, k, l\}$, the two ends of R_m are $\xi_{t_1}(m)$ and z_m for each $m \in \{i, j, k, l\}$, and $\{y_i, y_j, y_k, y_l\} = \{\xi_{t_2}(i), \xi_{t_2}(j), \xi_{t_2}(k), \xi_{t_2}(l)\}$, $\{z_i, z_j, z_k, z_l\} = \{\xi_{t_3}(i), \xi_{t_3}(j), \xi_{t_3}(k), \xi_{t_3}(l)\}$.

We say that t_0 *has property* A^0_{ijkl} *in* η if there exist four disjoint tripods L_i, L_j, L_k, L_l in G' such that for each $m \in \{i, j, k, l\}$, the tripod L_m has feet $\xi_{t_1}(m), \xi_{t_2}(m_2), \xi_{t_3}(m_3)$ for some $m_2, m_3 \in \{i, j, k, l\}$.

We say that t_0 *has property* A^1_{ijkl} *in* η if there exist vertices $v_{x,y}$ for all $x \in \{i, j, k, l\}$, $y \in \{2, 3\}$, and tripods L_i, L_j, L_k, L_l in G' with centers c_i, c_j, c_k, c_l such that:

- for each $y \in \{2, 3\}$, $\{v_{i,y}, v_{j,y}, v_{k,y}, v_{l,y}\} = \{\xi_{t_y}(i), \xi_{t_y}(j), \xi_{t_y}(k), \xi_{t_y}(l)\}$
- for each $m \in \{i, j, k, l\}$, L_m has feet $\xi_{t_1}(m)$, $v_{m,2}, v_{m,3}$
- $L_i \cap L_l = c_i L_i v_{i,3} \cap c_l L_l v_{l,2}$ and it is a path that does not contain c_i, c_l. Let v_h be the vertex of this path that is closest to c_h for $h \in \{i, l\}$
- $L_j \cap L_k = c_j L_j v_{j,3} \cap c_k L_k v_{k,2}$ and it is the null graph or a path that does not contain c_j, c_k

- $V(L_{h_1} \cap L_{h_2}) \subseteq V(c_{h_1} L_{h_1} v_{h_1}) - \{c_{h_1}, v_{h_1}\}$ for all $h_1 \in \{i, l\}$ and $h_2 \in \{j, k\}$

- the paths $\xi_{t_1}(m) L_m v_{m,2}$ for all $m \in \{i, j, k, l\}$ are disjoint and the paths $\xi_{t_1}(m) L_m v_{m,3}$ for all $m \in \{i, j, k, l\}$ are disjoint.

  See Figure 5.1(a).

We say that t_0 *has property* A^{1a}_{ijkl} *in* η if t_0 has property A^1_{ijkl} with $V(L_j) \cap V(L_k) = \emptyset$. We say that t_0 *has property* A^{1b}_{ijkl} *in* η if t_0 has property A^1_{ijkl} with $V(L_j) \cap V(L_k) \neq \emptyset$.

If t_0 has one of the properties above, we say that t_0 has that property *with ordered feet* if for all $h \in \{i, j, k, l\}$, L_h has feet $\xi_{t_1}(h), \xi_{t_2}(h), \xi_{t_3}(h)$.

We say that t_0 *has property* A^2_{ijkl} *in* η if there exist vertices $v_{x,y}$ for all $x \in \{i, j, k, l\}$, $y \in \{2, 3\}$, and tripods L_j, L_k, L_l in G' with centers c_j, c_k, c_l and disjoint paths L_i, R_i such that:

- for each $y \in \{2, 3\}$, $\{v_{i,y}, v_{j,y}, v_{k,y}, v_{l,y}\} = \{\xi_{t_y}(i), \xi_{t_y}(j), \xi_{t_y}(k), \xi_{t_y}(l)\}$

- for each $m \in \{j, k, l\}$, L_m has feet $\xi_{t_1}(m)$, $v_{m,2}, v_{m,3}$

- L_i has ends $\xi_{t_1}(i)$ and $v_{i,2}$ and R_i has $v_{i,3}$ as one end and c_i as the other end, where $c_i \in V(c_l L_l v_{l,2}) - \{c_l\}$

- L_i is disjoint from $L_j \cup L_k \cup L_l \cup R_i$ and R_i is internally disjoint from $L_j \cup L_k \cup L_l$

- $L_j \cap L_k = c_j L_j v_{j,3} \cap c_k L_k v_{k,2}$ and it is the null graph or a path that does not contain c_j, c_k

- $V(L_h \cap L_l) \subseteq (V(c_h L_h v_{h,3}) \cap V(c_l L_l c_i)) - \{c_i, c_h, c_l\}$ for all $h \in \{j, k\}$.

  See Figure 5.1(b).

We say that t_0 *has property* A^3_{ijkl} *in* η if there exist vertices $v_{x,y}$ for all $x \in \{i, j, k, l\}$, $y \in \{2, 3\}$, and tripods L_j, L_k, L_l in G' with centers c_j, c_k, c_l and disjoint paths L_i, R_i such that:

- for each $y \in \{2, 3\}$, $\{v_{i,y}, v_{j,y}, v_{k,y}, v_{l,y}\} = \{\xi_{t_y}(i), \xi_{t_y}(j), \xi_{t_y}(k), \xi_{t_y}(l)\}$

- for each $m \in \{j, k, l\}$, L_m has feet $\xi_{t_1}(m)$, $v_{m,2}, v_{m,3}$
- L_i has ends $\xi_{t_1}(i)$ and $v_{i,2}$ and R_i has $v_{i,3}$ as one end and c_i as the other end, where $c_i \in V(c_k L_k v_{k,2}) - \{c_k\}$
- L_i is disjoint from $L_j \cup L_k \cup L_l \cup R_i$ and R_i is internally disjoint from $L_j \cup L_k \cup \xi_{t_1}(l) L_l \xi_{t_3}(l)$
- $L_j \cap L_k = c_j L_j v_{j,3} \cap c_k L_k c_i$ and it is the null graph or a path that does not contain c_i, c_j, c_k
- $L_j \cap L_l = c_j L_j v_{j,3} \cap c_l L_l v_{l,2}$ and it is a path P_1 that does not contain c_j, c_l
- $R_i \cap c_l L_l v_{l,2}$ is the null graph or a path P_2 that does not contain c_i, c_l
- $L_k \cap L_l = c_k L_k v_{k,3} \cap c_l L_l v_{l,2}$ and it is the null graph or a path P_3 that does not contain c_k, c_l
- $v_{l,2}, P_1, P_2, P_3, c_l$ lie on $v_{l,2} L_l c_l$ in that order.

  See Figure 5.1(c).

We say that t_0 *has property* A^4_{ijkl} *in* η if there exist vertices $v_{x,y}$ for all $x \in \{i, j, k, l\}$, $y \in \{2, 3\}$, and tripods L_j, L_k, L_l in G' with centers c_j, c_k, c_l and disjoint paths L_i, R_i such that:

- for each $y \in \{2, 3\}$, $\{v_{i,y}, v_{j,y}, v_{k,y}, v_{l,y}\} = \{\xi_{t_y}(i), \xi_{t_y}(j), \xi_{t_y}(k), \xi_{t_y}(l)\}$
- for each $m \in \{j, k, l\}$, L_m has feet $\xi_{t_1}(m)$, $v_{m,2}, v_{m,3}$
- L_i has ends $\xi_{t_1}(i)$ and $v_{i,2}$ and R_i has $v_{i,3}$ as one end and c_i as the other end, where $c_i \in V(c_k L_k v_{k,2}) - \{c_k\}$
- L_i is disjoint from $L_j \cup L_k \cup L_l \cup R_i$ and R_i is internally disjoint from $L_j \cup L_k \cup L_l$
- $L_j \cap L_k = c_j L_j v_{j,3} \cap c_k L_k c_i$ and it is a path that does not contain c_i, c_j, c_k

- $L_j \cap L_l = \emptyset$

- $L_k \cap L_l = c_k L_k v_{k,3} \cap c_l L_l v_{l,2}$ and it is a path that does not contain c_k, c_l.

  See Figure 5.1(d).

If t_0 has property A^2_{ijkl}, A^3_{ijkl} or A^4_{ijkl}, we say that t_0 has that property *with ordered left-feet* if for all $h \in \{j, k, l\}$, L_h has foot $\xi_{t_2}(h)$ in $X_{\eta(t_2)}$.

If t_0 has one of the properties above, we will denote L_i, L_j, L_k, L_l, R_i by $L_i(t_0), L_j(t_0), L_k(t_0), L_l(t_0), R_i(t_0)$ and c_i, c_j, c_k, c_l by $c_i(t_0), c_j(t_0), c_k(t_0), c_l(t_0)$ when we want to emphasize they are in the η-torso at the major vertex t_0.

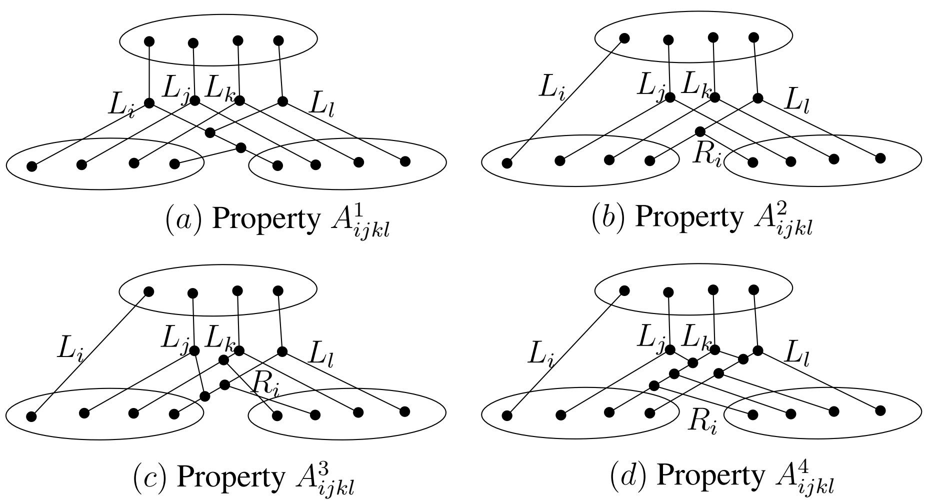


Figure 5.1: Properties A^m_{ijkl} for $m \in \{1, 2, 3, 4\}$.

Let A_{t_0} and B_{t_0} be the confinement sets for η at t_0. We say that t_0 *has property B in* η if s is even, A_{t_0} and B_{t_0} are disjoint and both have size $s/2$, and there exist disjoint paths $R_1, R_2, \ldots, R_{3s/2}$ in G', disjoint paths $Y_1, Y_2, ..., Y_{s/2}$ and $Z_1, Z_2, ..., Z_{s/2}$ internally disjoint from G', and a bijective function g from A_{t_0} to B_{t_0} in such a way that

- each R_i is a subpath of both the left specified t_0-linkage and the right specified t_0-linkage,

- for $i \in A_{t_0}$, the path R_i has ends $\xi_{t_1}(i)$ and $\xi_{t_2}(i)$,

- for $i \in B_{t_0}$ the path R_i has ends $\xi_{t_1}(i)$ and $\xi_{t_3}(i)$,
- for $i = s+1, s+2, \ldots, 3s/2$ the path R_i has one end $\xi_{t_2}(g(k))$ and the other end $\xi_{t_3}(k)$ for some $k \in A_{t_0}$,
- for $i \in A_{t_0}$, the path Y_i has ends $\xi_{t_2}(i)$ and $\xi_{t_2}(g(i))$, and
- for $i \in A_{t_0}$, the path Z_i has ends $\xi_{t_3}(i)$ and $\xi_{t_3}(g(i))$.

We say that t_0 *has property* B_{ijkl} *in* η if there exist pairwise disjoint paths $R_i, R_j, R_k, R_l, R_{ij}, R_{kl}$ and a path R in G' and disjoint paths Y_i, Y_k, Z_i, Z_k internally disjoint from G' such that

- the ends of R_h are $\xi_{t_1}(h)$ and $\xi_{t_2}(h)$ for $h \in \{i, k\}$
- the ends of R_h are $\xi_{t_1}(h)$ and $\xi_{t_3}(h)$ for $h \in \{j, l\}$
- the ends of R_{ij} are $\xi_{t_2}(j)$ and $\xi_{t_3}(i)$
- the ends of R_{kl} are $\xi_{t_2}(l)$ and $\xi_{t_3}(k)$
- the ends of Y_i are $\xi_{t_2}(i)$ and $\xi_{t_2}(j)$
- the ends of Y_k are $\xi_{t_2}(k)$ and $\xi_{t_2}(l)$
- the ends of Z_i are $\xi_{t_3}(i)$ and $\xi_{t_3}(j)$
- the ends of Z_k are $\xi_{t_3}(k)$ and $\xi_{t_3}(l)$
- R is internally disjoint from the remaining paths and connects two of the three paths R_i, R_j and R_{ij}.

We will denote these paths as $R_i(t_0), R_j(t_0), R_k(t_0), R_l(t_0), R_{ij}(t_0), R_{kl}(t_0), R(t_0), Y_i(t_0), Y_k(t_0), Z_i(t_0), Z_k(t_0)$ when we want to emphasize they are in the η-torso at the major vertex t_0.

We say that t_0 *has property* B^1_{ijkl} *in* η if there exist tripods L_i, L_j and disjoint paths R_k, R_l, R_{kl} in G' such that

- the ends of R_k are $\xi_{t_1}(k)$ and $\xi_{t_2}(k)$
- the ends of R_l are $\xi_{t_1}(l)$ and $\xi_{t_3}(l)$
- the ends of R_{kl} are $\xi_{t_2}(l)$ and $\xi_{t_3}(k)$
- the feet of L_h are $\xi_{t_1}(h), \xi_{t_2}(h), \xi_{t_3}(h)$ for $h \in \{i, j\}$
- R_m is disjoint from L_h for all $h \in \{i, j\}$ and $m \in \{k, l, kl\}$
- $L_i \cap L_j = c_i L_i \xi_{t_3}(i) \cap c_j L_j \xi_{t_2}(j)$ and it is a path that does not contain c_i, c_j, where c_h is the center of L_h for all $h \in \{i, j\}$.

We will denote these paths and tripods as $L_i(t_0), L_j(t_0), R_k(t_0), R_l(t_0), R_{kl}(t_0)$ when we want to emphasize they are in the η-torso at the major vertex t_0.

Lemma 5.1.1. *Let (T, X) be a tree-decomposition of a graph G. Let $\eta : T_1 \hookrightarrow T$ be an ordered cascade in (T, X) with orderings ξ_t of height one and size $s + |I|$, where I is the common intersection set. Let t_0 be the major vertex in T_1, and let $i, j, k, l \in \{1, 2, \ldots, s\}$ be distinct. If t_0 has property A_{ijkl} in η, then t_0 has property $A^m_{i'j'k'l'}$ in η for some $m \in \{0, 2, 3, 4\}$ and i', j', k', l' such that $\{i', j', k', l'\} = \{i, j, k, l\}$.*

Proof. Assume the trinity at t_0 is (t_1, t_2, t_3). As in the definition of property A_{ijkl}, in the η-torso at t_0 there exist disjoint paths L_i, L_j, L_k, L_l such that L_m has ends $\xi_{t_1}(m)$ and y_m for all $m \in \{i, j, k, l\}$ and there exist disjoint paths R_i, R_j, R_k, R_l such that R_m has ends $\xi_{t_1}(m)$ and z_m for all $m \in \{i, j, k, l\}$, $\{y_i, y_j, y_k, y_l\} = \{\xi_{t_2}(i), \xi_{t_2}(j), \xi_{t_2}(k), \xi_{t_2}(l)\}$, and $\{z_i, z_j, z_k, z_l\} = \{\xi_{t_3}(i), \xi_{t_3}(j), \xi_{t_3}(k), \xi_{t_3}(l)\}$. Let $x_m = \xi_{t_1}(m)$ for all $m \in \{i, j, k, l\}$. Among all the possible choices of such paths, choose the one such that $M = |\bigcup_m [E(L_m) \cup E(R_m)]|$ is minimal. Assume from z_i, z_j, z_k, z_l the paths R_i, R_j, R_k, R_l first meet $\bigcup_m L_m$ at a, b, c, d, respectively. We will use the following two facts that have the same proofs as Claims 4.1.1.1 and 4.1.1.2, respectively.

Claim 5.1.1.1. *Let $m, n \in \{i, j, k, l\}$. Assume R_m meets L_n at a vertex v. Then from v, after departing from the path L_n, vR_mx_m must meet L_h before L_n for some $h \in \{i, j, k, l\} - \{n\}$.*

Claim 5.1.1.2. *Let $m, n, h_1, h_2 \in \{i, j, k, l\}$ where $m \neq n$. Let P_1 be a subpath of R_{h_1} with two ends v_1, w_1 such that $v_1 \in V(L_m), w_1 \in V(L_n)$ and P_2 be a subpath of R_{h_2} with two ends v_2, w_2 such that $v_2 \in V(L_m), w_2 \in V(L_n)$. Assume P_1, P_2 are internally disjoint from $L_i \cup L_j \cup L_k \cup L_l$ and P_1 is disjoint from P_2. Assume $v_1 \in V(v_2L_my_m)$. Then $w_1 \in V(w_2L_ny_n)$.*

Back to the proof of the lemma, if no two of a, b, c, d lie on the same path in L_i, L_j, L_k, L_l, then it is clear that t_0 has property A^0_{ijkl} in η. So consider three cases:

Case 1: all of a, b, c, d lie on some path in L_i, L_j, L_k, L_l. Without loss of generality, assume they all lie on L_l such that y_l, a, b, c, d, x_l lie on L_l in that order. Assume from a, b, c the paths $aL_ix_i, bL_jx_j, cL_kx_k$ depart from L_l at a_1, b_1, c_1, respectively. By Claim 5.1.1.1, from a_1, b_1, c_1 the paths $aL_ix_i, bL_jx_j, cL_kx_k$ cannot meet L_l again before $L_i \cup L_j \cup L_k$. So assume from a_1, b_1, c_1 the paths $a_1L_ix_i, b_1L_jx_j, c_1L_kx_k$ meet $L_i \cup L_j \cup L_k$ at a_2, b_2, c_2, respectively. Without loss of generality assume $c_2 \in V(L_k)$. If $b_2 \in V(L_h)$ for some $h \in \{i, j\}$, then t_0 has property A^2_{ijkl} or A^2_{jikl}. So assume $b_2 \in V(L_k)$. Then there must be a subpath P of $b_2R_jx_j$ with two ends x, y such that P is internally disjoint from $\bigcup_{m \in \{i,j,k,l\}} L_m$ and $x \in V(L_h)$ for some $h \in \{i, j\}$ and $y \in V(L_k \cup L_l)$. We may assume that $x \in V(L_j)$. By Claim 5.1.1.2, $b_2 \in V(c_2L_ky_k)$. By Claim 5.1.1.2 again, the path P may be chosen so that $y \in V(c_2L_ky_k \cup c_1L_ly_l)$. If $y \in V(c_2L_ky_k)$ or $y \in V(c_1L_la)$, then t_0 has property A^2_{ijkl}. So assume $y \in V(aL_ly_l)$. Then t_0 has property A^3_{ijkl}.

Case 2: three of a, b, c, d lie on some path in L_i, L_j, L_k, L_l. Without loss of generality, assume that a, b, c lie on L_l and d lies on L_k such that y_l, a, b, c, x_l lie on L_l in that order. Assume from a, b the paths aL_ix_i, bL_jx_j depart from L_l at a_1, b_1, respectively, and from a_1, b_1 the paths $a_1L_ix_i, b_1L_jx_j$ first meet $L_i \cup L_j \cup L_k$ at a_2, b_2, respectively. If $b_2 \in V(L_i)$, then t_0 has property A^2_{jikl}. If $b_2 \in V(L_j)$, then t_0 has property A^2_{ijkl}. The remaining case is when $b_2 \in V(L_k)$. Assume $b_2 \in V(x_k L_k d)$. Then there must be a subpath P of $dR_l x_l$ with two ends x, y such that P is internally disjoint from $\bigcup_{m\in\{i,j,k,l\}} L_m$ and $x \in V(L_h)$ for some $h \in \{i, j\}$ and $y \in V(L_k \cup L_l)$. We may assume that $x \in V(L_j)$. By Claim 5.1.1.2, the path P may be chosen so that $y \in V(b_2 L_k y_k \cup b_1 L_l y_l)$. If $y \in V(b_2 L_k y_k)$, then t_0 has property A^2_{ijkl}. If $y \in V(b_1 L_l y_l)$, then t_0 has property A^3_{ijkl}. The case $b_2 \in V(dL_k y_k)$ is similar, but we will choose P as a subpath of $aR_i x_i$ instead of $dR_l x_l$.

Case 3: two of a, b, c, d lie on some path in L_i, L_j, L_k, L_l. Without loss of generality, assume that a, b lie on L_l such that y_l, a, b, x_l lie on L_l in that order. If c, d do not lie on the same path in L_i, L_j, L_k, L_l, without loss of generality assume c lies on L_k and d lies on L_j. Then t_0 has property A^2_{ijkl}. Therefore assume c, d lie on the same path L_k such that y_k, c, d, x_k lie on L_k in that order. Assume from a, c the paths $aL_i x_i, cL_k x_k$ depart from L_l, L_k at a_1, c_1, respectively, and from a_1, c_1 the paths $a_1 L_i x_i, c_1 L_k x_k$ first meet $L_i \cup L_j \cup L_k, L_i \cup L_k \cup L_l$ at a_2, c_2, respectively. If $a_2 \in V(L_i \cup L_j)$, then t_0 has property A^2_{ijlk} or A^2_{jilk}. If $c_2 \in V(L_i \cup L_j)$, then t_0 has property A^2_{ijkl} or A^2_{jikl}. Hence assume $a_2 \in V(L_k)$ and $c_2 \in V(L_l)$. Without loss of generality, assume $a_2 \in V(x_k L_k c)$ because if $a_2 \in V(cL_k y_k)$ then $c_2 \in V(x_l L_l a)$ by Claim 5.1.1.2. Then there exists a subpath P of $cR_k x_k$ with two ends x, y such that P is internally disjoint from $\bigcup_{m\in\{i,j,k,l\}} L_m$ and $x \in V(L_h)$ for some $h \in \{i, j\}$ and $y \in V(L_k \cup L_l)$. We may assume that $x \in V(L_j)$. Because $a_2 \in V(x_k L_k c)$, by Claim 5.1.1.2, the path P may be chosen so that $y \in V(a_2 L_k y_k \cup a_1 L_l y_l)$. If $y \in V(dL_k y_k)$, then t_0 has property A^2_{ijkl}. If $y \in V(a_1 L_l y_l)$, then t_0 has property A^2_{ijlk}. The remaining case is when $a_2 \in V(x_k L_k d)$ and $y \in V(a_2 L_k d)$, then t_0 has property A^4_{ijkl} with three tripods $L_j \cup P \cup yL_k d \cup dR_l z_l$, $L_k \cup a_2 R_i z_i$, and $L_l \cup bR_j z_j$ and two paths L_i and $cR_k z_k$. $\square$

5.2 Main lemma

Lemma 5.2.1. *Let $s \geq 4$ be an integer. Let (T, X) be a tree decomposition of a graph G satisfying $(W6)$, and let $\eta : T_7 \hookrightarrow T$ be a regular cascade in (T, X) of size $|I| + s$ with specified linkages that are minimal, where I is the common intersection set of η. Then either there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one such that in η' the unique major vertex of T_1 has property A_{ijkl} for some distinct integers $i, j, k, l \in \{1, 2, \ldots, s\}$ or the major root of T_7 has property B in η.*

Proof. We will either construct a weakly monotone homeomorphic embedding $\gamma : T_1 \hookrightarrow T_7$ such that in $\eta' = \eta \circ \gamma$ the major root of T_1 will have property A_{ijkl} for some distinct integers $i, j, k, l \in \{1, 2, \ldots, s\}$, or establish that the major root of T_7 has property B in η.

Since η is regular, there exist sets A, B, C, D as in the definition of a regular cascade. Let t_0 be the unique major vertex of T_1 and let (t_1, t_2, t_3) be its trinity. Let u_0 be the major root of T_7 and let (v_1, v_2, v_3) to be its trinity. Let u_1, u_2 be the major vertices of T_7 of height one such that u_1 is adjacent to v_2 and u_2 is adjacent to v_3. Let (v_2, v_4, v_5) be the trinity at u_1 and (v_3, v_6, v_7) be the trinity at u_2. Let u_3, u_4 be the major vertices of T_7 of height two such that u_3 is adjacent to v_4 and u_4 is adjacent to v_6. Let (v_4, v_8, v_9) be the trinity at u_3 and (v_6, v_{10}, v_{11}) be the trinity at u_4.

Let us recall that for a major vertex u of T_7 we denote the paths in the specified left u-linkage by $P_i(u)$ and the paths in the specified right u-linkage by $Q_i(u)$. If there exist three distinct integers $i, j, k, l \in A \cap B$, then the paths $P_h(u_0)$ and $Q_h(u_0)$ for $h \in \{i, j, k, l\}$ show that u_0 has property A_{ijkl} in η. Let $\gamma : T_1 \hookrightarrow T_7$ be the homeomorphic embedding that maps t_0, t_1, t_2, t_3 to u_0, v_1, v_2, v_3, respectively. Then $\eta' = \eta \circ \gamma$ is as desired. We may therefore assume that $|A \cap B| \leq 3$.

For $i \in \{1, 2, \ldots, s\} - A$ the path $P_i(u_0)$ exits and re-enters the η-torso at u_0, and it does so through two distinct vertices of $X_{\eta(v_3)} - I$. But $|X_{\eta(v_3)} - I| = s$, hence $|A| \geq s/2$. Similarly $|B| \geq s/2$. Let a be a major vertex with trinity (a_1, a_2, a_3). The set C includes an element of the form (i, l, m), which means that the vertices $\xi_{a_1}(i), \xi_{a_3}(l), \xi_{a_3}(m), \xi_{a_2}(i)$ appear on the path $P_i(a)$ in the order listed. Let $l_i := l, m_i := m$, $x_i(a) := \xi_{a_3}(l)$, $y_i(a) := \xi_{a_3}(m)$, $X_i(a) := \xi_{a_1}(i)P_i(a)x_i(a)$ and $Y_i(a) := y_i(a)P_i(a)\xi_{a_2}(i)$. Thus $X_i(a)$ and $Y_i(a)$ are subpaths of the η-torso at a. Similarly, if $i \in \{1, 2, .., s\} - B$, then the set D includes an element of the form (i, n, r), which means that the vertices $\xi_{a_1}(i), \xi_{a_2}(n), \xi_{a_2}(r), \xi_{a_3}(i)$ appear on the path $Q_i(a)$ in the order listed. Let $n_i := n, r_i := r$, $w_i(a) := \xi_{a_2}(n)$, $z_i(a) := \xi_{a_2}(r)$, $W_i(a) := \xi_{a_1}(i)Q_i(a)w_i(a)$ and $Z_i(a) := z_i(a)Q_i(a)\xi_{a_3}(i)$.

For any minor vertex w in T_7, let S_w be the vertex set of the outer graph at w.

Claim 5.2.1.1. *Assume $|A \cap B| \geq 1$. Let $a, b \in \{1, 2, ..., s\}$ be distinct. If w is a minor vertex of height at most three in T_7, then there exists a path P between $\xi_w(a)$ and $\xi_w(b)$ in G and a descendant v of w such that v is a minor vertex of T_7 at height at most five and the internal vertices of P are in $S_w - S_v$.*

To prove the claim let the child of w be z, and the trinity at z be (w, w_1, w_2). If $a, b \in A$, let $v = w_2$ and P be the union of M_1 and M_2 and a W6-path in the outer graph at w_1 joining their ends, where $M_1 = P_a(z)$ and $M_2 = P_b(z)$. If $a, b \in B$, let $v = w_1$ and P be the union of M_1 and M_2 and a W6-path in the outer graph at w_2 joining their ends, where $M_1 = Q_a(z)$ and $M_2 = Q_b(z)$. If one of a, b is not in $A \cup B$, without loss of generality, assume $a \notin A \cup B$. If $b \notin A \cup B$, then let $v = w_2$ and P be the union of M_1 and M_2 and a W6-path in the outer graph at w_1 joining their ends, where $M_1 = W_a(z)$ and $M_2 = W_b(z)$. If $b \in A \cup B$, without loss of generality, assume $b \in A$. Let P' be the union of M_1 and M_2 and a W6-path in the outer graph at w_1 joining their ends, where $M_1 = W_a(z)$ and $M_2 = P_b(z)$. Then let $v = w_2$ and P be a subpath with same ends as P'. Then the remaining case is when $a \in A - B$ and $b \in B - A$, or $a \in B - A$ and $b \in A - B$. Without loss of generality, assume $a \in A - B$ and $b \in B - A$. Let z_1 be the child of w_1 and its trinity be (w_1, w_3, w_4), and let z_2 be the child of w_3 and its trinity be (w_3, w_5, w_6). Let $j \in A \cap B$ and let $v = w_6$. Let $M_1 = P_b(z) \cup X_b(z_1)$, $M_2 = Q_j(z_1) \cup P_j(z_1) \cup P_j(z_2)$ and $M_3 = P_a(z) \cup P_a(z_1) \cup P_a(z_2)$. Let P' be the union of M_1, M_2, M_3, a $W6$-path in the outer graph at w_4 joining the ends of M_1 and M_2, and a $W6$-path in the outer graph at w_5 joining the ends of M_2 and M_3 by Lemma 3.4.2. Let P be a subpath with same ends as P', then v and P are as desired.

Back to the main lemma, we distinguish four main cases.

Main case 1: $|A \cap B| = 3$. Assume $A \cap B = \{i, j, l\}$. Assume $B - A = \emptyset$, then $B = \{i, j, l\}$. Let $k \in \{1, ..., s\} - B$. Let $\gamma(t_0) = u_0, \gamma(t_1) = v_1$.

Consider the following cases depending on n_k and r_k. If $n_k, r_k \in B$ (so they are also in A), let $\gamma(t_2) = v_4$, $E_h = P_h(u_1)$ for all $h \in \{i, j, k, l\}$, and let L be the union of $W_k(u_0) \cup Q_{n_k}(u_1)$ and $Q_{r_k}(u_1) \cup Z_k(u_0)$ and a $W6$-path in the outer graph at v_5 joining their ends by Lemma 3.4.2. If at least one of n_k, r_k is not in B, let $\gamma(t_2) = v_5$ and $E_h = Q_h(u_1)$ for all $h \in \{i, j, k, l\}$. If $n_k, r_k \notin B$, let L be the union of $W_k(u_0) \cup W_{n_k}(u_1)$ and $W_{r_k}(u_1) \cup Z_k(u_0)$ and a $W6$-path in the outer graph at v_4 joining their ends by Lemma 3.4.2. If $n_k \in B$ and $r_k \notin B$, let L be the union of $W_k(u_0) \cup P_{n_k}(u_1)$ and $W_{r_k}(u_1) \cup Z_k(u_0)$ and a $W6$-path in the outer graph at v_4 joining their ends by Lemma 3.4.2. If $n_k \notin B$ and $r_k \in B$, let L be the union of $W_k(u_0) \cup W_{n_k}(u_1)$ and $P_{r_k}(u_1) \cup Z_k(u_0)$ and a $W6$-path in the outer graph at v_4 joining their ends by Lemma 3.4.2.

If $k \in A$ then we let $\gamma(t_3) = v_3$, $F_h = \emptyset$ for all $h \in \{i, j, k, l\}$ and $R = P_k(u_0)$. If $k \notin A$ then we consider the following cases depending on l_k and m_k. If $l_k, m_k \in B$, let $\gamma(t_3) = v_6$, $F_h = P_h(u_2)$ for all $h \in \{i, j, k, l\}$, and let R be the union of $X_k(u_0) \cup Q_{l_k}(u_2)$ and $Q_{m_k}(u_2) \cup Y_k(u_0)$ and a $W6$-path in the outer graph at v_7 joining their ends by Lemma 3.4.2. If at least one of l_k, m_k is not in B, let $\gamma(t_3) = v_7$ and $F_h = Q_h(u_2)$ for all $h \in \{i, j, k, l\}$. If $l_k, m_k \notin B$, let R be the union of $X_k(u_0) \cup W_{l_k}(u_2)$ and $W_{m_k}(u_2) \cup Y_k(u_0)$ and a $W6$-path in the outer graph at v_6 joining their ends by Lemma 3.4.2. If $l_k \in B$ and $m_k \notin B$, let R be the union of $X_k(u_0) \cup P_{l_k}(u_2)$ and $W_{m_k}(u_2) \cup Y_k(u_0)$ and a $W6$-path in the outer graph at v_6 joining their ends by Lemma 3.4.2. If $l_k \notin B$ and $m_k \in B$, let R be the union of $X_k(u_0) \cup W_{l_k}(u_2)$ and $P_{m_k}(u_2) \cup Y_k(u_0)$ and a $W6$-path in the outer graph at v_6 joining their ends by Lemma 3.4.2.

Let L' be a subpath of L with the same ends and R' be a subpath of R with the same ends. Then the unique major vertex of T_1 has property A_{ijkl} in $\eta' = \eta \circ \gamma$ with the first 4-tuple of disjoint paths being $R' \cup E_k$ and $P_h(u_0) \cup E_h$ for all $h \in \{i, j, l\}$, and the second 4-tuple being $L' \cup F_k$ and $Q_h(u_0) \cup F_h$ for all $h \in \{i, j, l\}$.

Now assume $B - A \neq \emptyset$. Select an element $k \in B - A$. Let $\gamma(t_0) = u_0, \gamma(t_1) = v_1, \gamma(t_2) = v_2$. By Claim 5.2.1.1 for $a = x_k(u_0), b = y_k(u_0)$ and $w = v_3$, there exist a descendant v of v_3 and a path P between a and b such that internal vertices of P are in $S_{v_3} - S_v$. Let $\gamma(t_3) = v$ and F_h be the disjoint paths from $\xi_{v_3}(h)$ to $\xi_v(h)$ for all $h \in \{i, j, k, l\}$ constructed by specified linkages. Then the unique major vertex of T_1 has property A_{ijkl} in $\eta' = \eta \circ \gamma$ with the first 4-tuple of disjoint paths being $P_h(u_0)$ for all $h \in \{i, j, l\}$ and a path between $\xi_{v_1}(k)$ and $\xi_{v_2}(k)$ that is a subgraph of $X_k(u_0) \cup P \cup Y_k(u_0)$, and the second 4-tuple being $Q_h(u_0) \cup F_h$ for all $h \in \{i, j, k, l\}$.

Main case 2: $|A \cap B| = 2$. Assume $A \cap B = \{j, l\}$. First assume $A - B = \emptyset$, then $A = \{j, l\}$ and $s = 4$. Let $\gamma(t_0) = u_0, \gamma(t_1) = v_1$. Let distinct $i, k \in \{1, ..., s\} - A$.

If $|B| = 4$, let $\gamma(t_3) = v_6$, $F_h = P_h(u_2)$ for all $h \in \{i, j, k, l\}$, and $R_h = Q_{l_h}(u_2) \cup x_h(u_2)P_h(u_2)y_h(u_2) \cup Q_{m_h}(u_2)$ for $h \in \{i, k\}$. If $|B| = 3$, let $\gamma(t_3) = v_{10}$ and $F_h = P_h(u_2) \cup P_h(u_4)$ for all $h \in \{i, j, k, l\}$. Without loss of generality, assume $l_i \notin B$. Let $R_k = Q_{l_k}(u_2) \cup x_k(u_2)P_k(u_2)y_k(u_2) \cup Q_{m_k}(u_2)$ and $R_i = W_{l_i}(u_2) \cup P \cup Z_{l_i}(u_2) \cup x_i(u_2)P_i(u_2)y_i(u_2) \cup Q_{m_i}(u_2)$, where P is a walk that we are about to construct. If $n_{l_i}, r_{l_i} \in B$, let P be the union of $Q_{n_{l_i}}$ and $Q_{r_{l_i}}$ and a W6-path in the outer graph at v_{11} joining their ends. If exactly one of n_{l_i}, r_{l_i} is not in B, assume $n_{l_i} \notin B$. Then n_{l_i} is also not in A and $r_{l_i} \in B$. Then let P be the union of $X_{n_{l_i}}$ and $Q_{r_{l_i}}$ and a W6-path in the outer graph at v_{11} joining their ends. If $|B| = 2$, we consider two small cases. If there exist $h_1, h_2 \in \{i, k\}$ such that $l_{h_1}, m_{h_2} \in B$, then there exist $h_3 \in \{i, k\} - \{h_1\}, h_4 \in \{i, k\} - \{h_2\}$ such that $l_{h_3}, m_{h_4} \notin B$. In this case let $\gamma(t_3) = v_{10}$, $F_h = P_h(u_2) \cup P_h(u_4)$ for all $h \in \{i, j, k, l\}$, R_{h_1} be the union of $Q_{l_{h_1}}(u_2)$ and $Q_{m_{h_2}}(u_2)$ and a W6-path in the outer graph at v_7 joining their ends, and R_{h_3} is the union of $W_{l_{h_3}} \cup P_1$ and $W_{m_{h_4}} \cup P_2$ and a W6-path in the outer graph at v_{11} joining their ends, where P_1 and P_2 are paths that we are about to construct. If $n_{l_{h_3}} \in B$, let $P_1 = Q_{n_{l_{h_3}}}(u_4)$, else let $P_1 = X_{n_{l_{h_3}}}(u_4)$. If $n_{m_{h_4}} \in B$, let $P_2 = Q_{n_{m_{h_4}}}(u_4)$, else let $P_2 = X_{n_{m_{h_4}}}(u_4)$. The remaining case is when $l_i, l_k \in B$ or $m_i, m_k \in B$. Then $\bigcup_{h \notin B} W_h(u_2) \cup \bigcup_{h \notin B} Z_h(u_2) \cup \bigcup_{h \notin A} x_h(u_2)P_h(u_2)y_h(u_2) \cup \bigcup_{h \in B} Q_h(u_2)$ is a left u_2-linkage, a contradiction to the minimality of specified u_2-linkages.

If $|B| = 4$, let $\gamma(t_2) = v_2$, $E_h = \emptyset$ for all $h \in \{i,j,k,l\}$, and $L_h = Q_h(u_0)$ for all $h \in \{i,k\}$. If $|B| = 3$, assume $i \in B$ and $k \notin B$. If $n_k \in B$, let $P_1 = Q_{n_k}(u_1)$, else let $P_1 = X_{n_k}(u_1)$. If $r_k \in B$, let $P_2 = Q_{r_k}(u_1)$, else let $P_2 = X_{r_k}(u_1)$. Then let $\gamma(t_2) = v_4$, $E_h = P_h(u_1)$ for all $h \in \{i,j,k,l\}$, $L_i = Q_i(u_0)$, and L_k be the union of $W_k(u_0) \cup P_1$ and $Z_k(u_0) \cup P_2$ and a W6-path in the outer graph at v_5 joining their ends. If $|B| = 2$, we consider two small cases. If there exist $h_1, h_2 \in \{i,k\}$ such that $n_{h_1}, r_{h_2} \in B$, then there exist $h_3 \in \{i,k\} - \{h_1\}, h_4 \in \{i,k\} - \{h_2\}$ such that $n_{h_3}, r_{h_4} \notin B$. In this case let $\gamma(t_2) = v_8$, $E_h = P_h(u_1) \cup P_h(u_3)$ for all $h \in \{i,j,k,l\}$, L_{h_1} be the union of $W_{h_1}(u_0) \cup Q_{n_{h_1}}(u_1)$ and $Z_{h_2} \cup Q_{r_{h_2}}(u_1)$ and a W6-path in the outer graph at v_5 joining their ends, and L_{h_3} be the union of $W_{h_3}(u_0) \cup W_{n_{h_3}} \cup P_1$ and $Z_{h_4}(u_0) \cup W_{r_{h_4}} \cup P_2$ and a W6-path in the outer graph at v_9 joining their ends, where P_1 and P_2 are paths that we are about to construct. If $n_{n_{h_3}} \in B$, let $P_1 = Q_{n_{n_{h_3}}}(u_3)$, else let $P_1 = X_{n_{n_{h_3}}}(u_3)$. If $n_{r_{h_4}} \in B$, let $P_2 = Q_{n_{r_{h_4}}}(u_3)$, else let $P_2 = X_{n_{r_{h_4}}}(u_3)$. The remaining case is when $n_i, n_k \in B$ or $r_i, r_k \in B$. Then $\bigcup_{h \notin A} X_h(u_1) \cup \bigcup_{h \notin A} Y_h(u_1) \cup \bigcup_{h \notin B} n_h(u_1)Q_h(u_1)r_h(u_1) \cup \bigcup_{h \in A} P_h(u_1)$ is a right u_1-linkage, a contradiction to the minimality of specified u_1-linkages.

Let L'_h be a subpath of L_h with the same ends and R'_h be a subpath of R_h with the same ends for all $h \in \{i,k\}$. Then the unique major vertex of T_1 has property A_{ijkl} in $\eta' = \eta \circ \gamma$ with the first 4-tuple of disjoint paths being $P_h(u_0) \cup E_h$ for all $h \in \{j,l\}$ and $X_h(u_0) \cup R'_h \cup Y_h(u_0) \cup E_h$ for all $h \in \{i,k\}$, and the second 4-tuple being $Q_h(u_0) \cup F_h$ for all $h \in \{j,l\}$ and $L'_h \cup F_h$ for all $h \in \{i,k\}$.

Therefore we can assume $A - B \neq \emptyset$ and $B - A \neq \emptyset$. Let $i \in A - B$ and $k \in B - A$. Let $\gamma(t_0) = u_0, \gamma(t_1) = v_1$. By Claim 5.2.1.1 for $a = w_i(u_0), b = z_i(u_0)$ and $w = v_2$, there exist a descendant v of v_2 and a path P between a and b such that internal vertices of P are in $S_{v_2} - S_v$. Let $\gamma(t_2) = v$, $L = P$, and E_h be the disjoint paths from $\xi_{v_2}(h)$ to $\xi_v(h)$ for all $h \in \{i,j,k,l\}$ constructed by specified linkages. Also by Claim 5.2.1.1 for $a = x_k(u_0), b = y_k(u_0)$ and $w = v_3$, there exist a descendant v' of v_3 and a path P' between a and b such that internal vertices of P' are in $S_{v_3} - S_{v'}$. Let $\gamma(t_3) = v'$, $R = P'$, and F_h be the disjoint paths from $\xi_{v_3}(h)$ to $\xi_{v'}(h)$ for all $h \in \{i,j,k,l\}$ constructed by specified linkages. Then the unique major vertex of T_1 has property A_{ijkl} in $\eta' = \eta \circ \gamma$ with the first 4-tuple of disjoint paths being $P_h(u_0) \cup E_h$ for all $h \in \{i,j,l\}$ and $X_k(u_0) \cup R \cup Y_k(u_0) \cup E_k$, and the second 4-tuple being $Q_h(u_0) \cup F_h$ for all $h \in \{j,k,l\}$ and $W_i(u_0) \cup L \cup Z_i(u_0) \cup F_i$.

Main case 3: $|A \cap B| = 1$. Let j be the unique element of $A \cap B$. Notice that $A - B \neq \emptyset$. In fact, if $A - B = \emptyset$, then $|A| = 1$. So $2(s-1) \leq s$ and this means $s \leq 2$, a contradiction. Similarly, $B - A \neq \emptyset$. Therefore, we can let $i \in A - B$ and $k \in B - A$. Let $l \in \{1,2,...,s\} - \{i,j,k\}$. Let $\gamma(t_0) = u_0, \gamma(t_1) = v_1$. Let u_5 be the child of v_5 and let (v_5, v_{12}, v_{13}) be its trinity.

If $l \in B$, by Claim 5.2.1.1 for $a = w_i(u_0), b = z_i(u_0)$ and $w = v_2$, there exist a minor vertex v and a path P between a and b such that internal vertices of P are in $S_{v_2} - S_v$. Let $\gamma(t_2) = v$, E_h be the disjoint paths from $\xi_{v_2}(h)$ to $\xi_v(h)$ for all $h \in \{i,j,k,l\}$, $L_l = Q_l(u_0)$, and $L_i = W_i(u_0) \cup P \cup Z_i(u_0)$. For the rest of this paragraph and the next three paragraphs, assume $l \notin B$. Assume $n_i, r_i, n_l, r_l \in A$ or $n_i, r_i, n_l, r_l \in B$. Without loss of generality, assume $n_i, r_i, n_l, r_l \in A$. Then let $\gamma(t_2) = v_5$, $E_h = P_h(u_1)$ for all $h \in \{i,j,k,l\}$, and $L_h = W_h(u_0) \cup P_{n_h}(u_1) \cup w_h(u_1)Q_h(u_1)z_h(u_1) \cup P_{r_h}(u_1) \cup Z_h(u_0)$ for $h \in \{i,l\}$. Assume three of n_i, r_i, n_l, r_l are in A or in B. Without loss of generality, assume $n_i, r_i, n_l \in B$ and $r_l \notin B$. By Claim 5.2.1.1 for $a = w_{r_l}(u_1), b = z_{r_l}(u_1)$ and $w = v_4$, there exist a descendant v of v_4 and a path P between a and b such that internal vertices of P are in $S_{v_4} - S_v$. Then let $\gamma(t_2) = v$, E_h be the disjoint paths from $\xi_{v_2}(h)$ to $\xi_v(h)$ for all $h \in \{i,j,k,l\}$, and $L_h = W_h(u_0) \cup Q_{n_h}(u_1) \cup P_{n_h}(u_5) \cup w_h(u_5)Q_h(u_5)z_h(u_5) \cup P_{r_h}(u_5) \cup Q_h \cup Z_h(u_0)$ for all $h \in \{i,l\}$, where $Q_h = Q_{r_h}(u_1)$ if $h = i$ and $Q_h = W_{r_h}(u_1) \cup P \cup Z_{r_h}(u_1)$ if $h = l$.

See Figure 5.2.

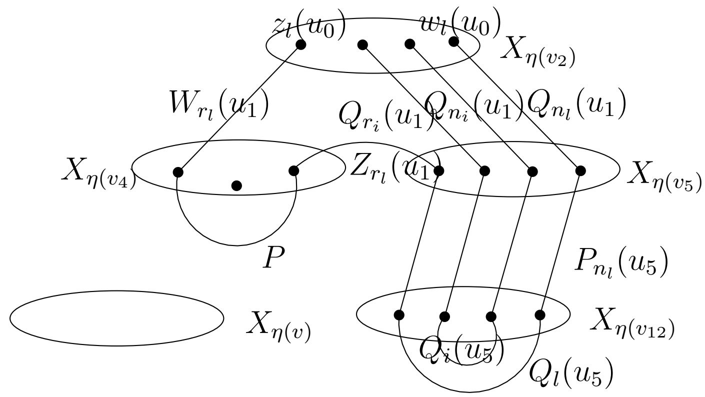


Figure 5.2: $n_i, r_i, n_l \in B$ and $r_l \notin B$.

Now assume exactly two of n_i, r_i, n_l, r_l are in A or exactly two of n_i, r_i, n_l, r_l are in B. Without loss of generality, assume exactly two of n_i, r_i, n_l, r_l are in A. If n_{h_1} and r_{h_2} are in A for some $h_1, h_2 \in \{i, l\}$, then there exist $h_3 \in \{i, l\} - \{h_1\}, h_4 \in \{i, l\} - \{h_2\}$ such that n_{h_3} and r_{h_4} are not in A. Then let $\gamma(t_2) = v_9$, $E_h = P_h(u_1) \cup Q_h(u_3)$ for all $h \in \{i, j, k, l\}$, L_{h_1} be the union of $W_{h_1}(u_0) \cup P_{n_{h_1}}(u_1) \cup P_{n_{h_1}}(u_3)$ and $Z_{h_2}(u_0) \cup P_{r_{h_2}}(u_1) \cup P_{r_{h_2}}(u_3)$ and a W6-path in the outer graph at v_8 joining their ends, and L_{h_3} be the union of $W_{h_3}(u_0) \cup X_{n_{h_3}}(u_1)$ and $Z_{h_4}(u_0) \cup X_{r_{h_4}}$ and a W6-path in the outer graph at v_5 joining their ends. The remaining case is when $n_i, n_l \in A$ or $n_i, n_l \in B$. Without loss of generality, assume $n_i, n_l \in A$ and $r_i, r_l \notin A$. Assume there exists a path M_1 in the η-torso at u_1 that connects $P_{n_i}(u_1) \cup P_{n_l}(u_1)$ and $P_{r_i}(u_1) \cup P_{r_l}(u_1)$ and is internally disjoint from $P_{n_i}(u_1) \cup P_{n_l}(u_1) \cup P_{r_i}(u_1) \cup P_{r_l}(u_1)$. Without loss of generality, assume M_1 connects $P_{n_i}(u_1)$ and $P_{r_i}(u_1)$. By Claim 5.2.1.1 for $a = \xi_{v_4}(n_l), b = \xi_{v_4}(r_l)$ and $w = v_4$, there exist a descendant v of v_4 and a path P between a and b such that internal vertices of P are in $S_{v_4} - S_v$. Let $\gamma(t_2) = v$, E_h be the disjoint paths from $\xi_{v_2}(h)$ to $\xi_v(h)$ for all $h \in \{i, j, k, l\}$, and $L_h = W_h(u_0) \cup P_{n_h}(u_1) \cup P_h \cup P_{r_h}(u_1) \cup Z_h(u_0)$ for all $h \in \{i, l\}$, where $P_h = M_1$ when $h = i$ and $P_h = P$ when $h = l$. See Figure 5.3.

Now assume there is no path in the η-torso at u_1 between $P_{n_i}(u_1) \cup P_{n_l}(u_1)$ and $P_{r_i}(u_1) \cup P_{r_l}(u_1)$. By Lemma 3.4.2, without loss of generality, assume in the outer graph

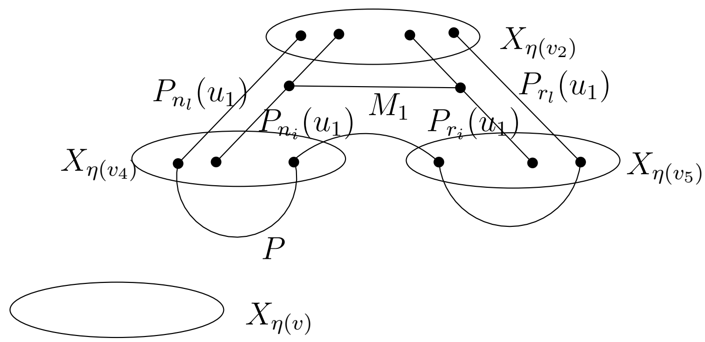


Figure 5.3: M_1 connects $P_{n_i}(u_1)$ and $P_{r_i}(u_1)$.

at v_5 there exists a path M_1 from $\xi_{v_5}(j)$ to $x_{r_i}(u_1)P_{r_i}(u_1)y_{r_i}(u_1)$ that is disjoint from $x_{r_l}(u_1)P_{r_l}(u_1)y_{r_l}(u_1)$, and assume in the outer graph at v_8 there exists a path M_2 from $\xi_{v_8}(j)$ to $w_i(u_3)Q_i(u_3)z_i(u_3)$ that is disjoint from $w_l(u_3)Q_l(u_3)z_l(u_3)$. If $Q_j(u_1)$ is not disjoint from $P_{n_i}(u_1) \cup P_{n_l}(u_1)$, then $Q_j(u_1)$ is disjoint from $P_{r_i}(u_1) \cup P_{r_l}(u_1)$. From $\xi_{v_5}(j)$, assume $Q_j(u_1)$ meets $P_{n_i}(u_1) \cup P_{n_l}(u_1)$ first at a vertex $x \in V(P_{n_i}(u_1))$. By Claim 5.2.1.1 for $a = \xi_{v_4}(n_l), b = \xi_{v_4}(r_l)$ and $w = v_4$, there exist a descendant v of v_4 and a path P between a and b such that internal vertices of P are in $S_{v_4} - S_v$. Then let $\gamma(t_2) = v$, E_h be the disjoint paths from $\xi_{v_2}(h)$ to $\xi_v(h)$ for all $h \in \{i, j, k, l\}$, and $L_h = W_h(u_0) \cup P_{n_h}(u_1) \cup P_h \cup P_{r_h}(u_1) \cup Z_h(u_0)$ for all $h \in \{i, l\}$, where $P_h = M_1 \cup \xi_{v_5}(j)Q_j(u_1)x$ when $h = i$ and $P_h = P$ when $h = l$. See Figure 5.4.

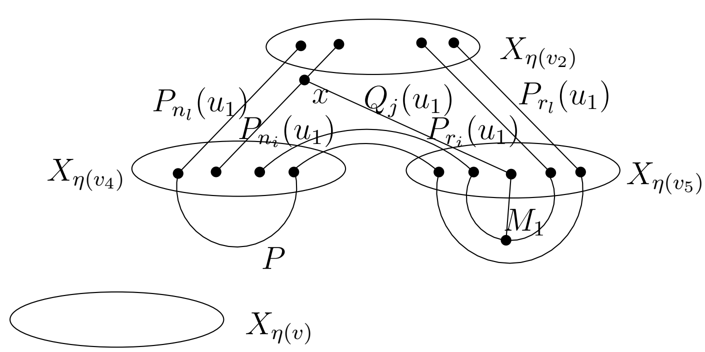


Figure 5.4: $j \in \{n_i, n_l\}$.

The remaining case is when $Q_j(u_1)$ is disjoint from $P_{n_i}(u_1) \cup P_{n_l}(u_1) \cup P_{r_i}(u_1) \cup P_{r_l}(u_1)$, or $Q_j(u_1)$ is disjoint from $P_{n_i}(u_1) \cup P_{n_l}(u_1)$ and not disjoint from $P_{r_i}(u_1) \cup P_{r_l}(u_1)$. This implies $j \notin \{n_i, n_l\}$. In the case $Q_j(u_1)$ is disjoint from $P_{n_i}(u_1) \cup P_{n_l}(u_1)$ and not disjoint from $P_{r_i}(u_1) \cup P_{r_l}(u_1)$, from $\xi_{v_2}(j)$ assume $Q_j(u_1)$ first meets $P_{r_i}(u_1) \cup P_{r_l}(u_1)$ at $y \in V(P_{r_i}(u_1))$. Then by Claim 5.2.1.1 for $a = x_{r_l}(u_3), b = y_{r_l}(u_3)$ and $w = v_9$, there exist a descendant v of v_9 and a path P between a and b such that internal vertices of P are in $S_{v_9} - S_v$. Then let $\gamma(t_2) = v$, E_h be the disjoint paths from $\xi_{v_2}(h)$ to $\xi_v(h)$ for all $h \in \{i, j, k, l\}$, and $L_h = W_h(u_0) \cup P_{n_h}(u_1) \cup P_{n_h}(u_3) \cup w_h(u_3)Q_h(u_3)z_h(u_3) \cup P_h \cup P_{r_h}(u_1) \cup Z_h(u_0)$ for all $h \in \{i, l\}$, where $P_i = M_1 \cup Q_j(u_1) \cup P_j(u_1) \cup P_j(u_3) \cup M_2$ if $Q_j(u_1)$ is disjoint from $P_{n_i}(u_1) \cup P_{n_l}(u_1) \cup P_{r_i}(u_1) \cup P_{r_l}(u_1)$, $P_i = yQ_j(u_1)\xi_{v_2}(j) \cup P_j(u_1) \cup P_j(u_3) \cup M_2$ if $Q_j(u_1)$ is disjoint from $P_{n_i}(u_1) \cup P_{n_l}(u_1)$ and not disjoint from $P_{r_i}(u_1) \cup P_{r_l}(u_1)$, and $P_l = Y_{r_l}(u_3) \cup P \cup X_{r_l}(u_3)$. See Figure 5.5.

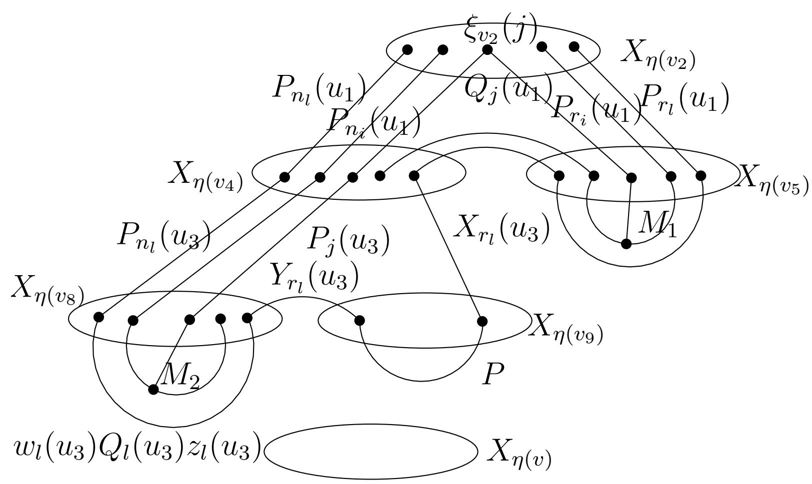


Figure 5.5: $Q_j(u_1)$ is disjoint from $P_{n_i}(u_1) \cup P_{n_l}(u_1) \cup P_{r_i}(u_1) \cup P_{r_l}(u_1)$.

Similarly, repeat the argument above by replacing n_i, n_l, r_i, r_l by l_k, l_l, m_k, m_l, we get $\gamma(t_3)$, F_h for all $h \in \{i, j, k, l\}$, and R_h for $h \in \{k, l\}$ such that $\gamma(t_3)$ is a descendant of v_3, F_h are disjoint paths from $\xi_{v_3}(h)$ to $\xi_{\gamma(t_3)}(h)$, and internal vertices of R_h are not in $S_{\gamma(t_3)}$.

Let L'_h be a subpath of L_h with the same ends for all $h \in \{i, l\}$ and R'_h be a subpath of R_h with the same ends for all $h \in \{k, l\}$. Then the unique major vertex of T_1 has property A_{ijkl} in $\eta' = \eta \circ \gamma$ with the first 4-tuple of disjoint paths being $P_h(u_0) \cup E_h$ for all $h \in \{i, j\}$

and $R'_h \cup E_h$ for all $h \in \{k, l\}$, and the second 4-tuple being $Q_h(u_0) \cup F_h$ for all $h \in \{j, k\}$ and $L'_h \cup F_h$ for all $h \in \{i, l\}$.

Main case 4: $A \cap B = \emptyset$. It follows that s is even and $|A| = |B| = s/2$. Assume as a case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$ and for some $k \in A$, $n_k, r_k \in A$ or $n_k, r_k \in B$. But the integers l_i, m_i are pairwise distinct, and so if $l_i, m_i \in A$, then there exists $j \in B$ such that $l_j, m_j \in B$, and similarly if $l_i, m_i \in B$. If $n_k, r_k \in A$, then there exists $l \in A$ such that $n_l, r_l \in B$, and similarly if $n_k, r_k \in B$. We may therefore assume that there exist $k, l \in A$ and $i, j \in B$ such that $n_k, r_k, l_i, m_i \in A$ and $n_l, r_l, l_j, m_j \in B$. We let γ map t_0, t_1, t_2, t_3 to u_0, v_1, v_9, v_{11}, respectively, and we will prove that t_0 has property A_{ijkl} in η'. To that end we need to construct two 4-tuples of disjoint paths. The first two paths of the first 4-tuple are $Q_i(u_0) \cup P_i(u_2) \cup Q_i(u_4)$ and $Q_j(u_0) \cup P_j(u_2) \cup Q_j(u_4)$. The third path of the first 4-tuple is the union of $W_k(u_0) \cup P_{n_k}(u_1) \cup P_{n_k}(u_3)$ and $P_{r_k}(u_3) \cup P_{r_k}(u_1) \cup Z_k(u_0) \cup P_k(u_2) \cup Q_k(u_4)$ and a suitable W6-path in the outer graph at v_8 joining their ends by Lemma 3.4.2. The fourth path of the first 4-tuple is the union of $W_l(u_0) \cup X_{n_l}(u_1)$ and $X_{r_l}(u_1) \cup Z_l(u_0) \cup P_l(u_2) \cup Q_l(u_4)$ and a suitable W6-path in the outer graph at v_5 joining their ends by Lemma 3.4.2. The first two paths of the second 4-tuple is $P_k(u_0) \cup P_k(u_1) \cup Q_k(u_3)$ and $P_l(u_0) \cup P_l(u_1) \cup Q_l(u_3)$. The third path of the second 4-tuple is the union of $X_i(u_0) \cup P_{l_i}(u_2) \cup P_{l_i}(u_4)$ and $Q_i(u_3) \cup P_i(u_1) \cup Y_i(u_0) \cup P_{m_i}(u_2) \cup P_{m_i}(u_4)$ and a suitable W6-path in the outer graph at v_{10} joining their ends by Lemma 3.4.2. The fourth path of the second 4-tuple is the union of $X_j(u_0) \cup X_{l_j}(u_2)$ and $Q_j(u_3) \cup P_j(u_1) \cup Y_j(u_0) \cup X_{m_j}(u_2)$ and a suitable W6-path in the outer a graph at v_7 joining their ends by Lemma 3.4.2. This completes the case that for some integer $i \in B$ either $l_i, m_i \in A$ or $l_i, m_i \in B$ and for some integer $k \in A$ either $n_k, r_k \in A$ or $n_k, r_k \in B$.

We may therefore assume that for every $i \in B$ one of l_i, m_i belongs to A and the other belongs to B, or for every $k \in A$ one of n_k, r_k belongs to A and the other belongs to B. Without loss of generality, assume that for every $i \in B$ one of l_i, m_i belongs to A and the other belongs to B. For every $i \in B$ a subpath of $P_i(u_0)$ joins $\xi_{v_3}(l_i)$ to $\xi_{v_3}(m_i)$

in the outer graph at v_3 and is disjoint from the η-torso at u_0, except for its ends. Let J be the union of these subpaths; then J is a linkage from $\{\xi_{v_3}(i) : i \in A\}$ to $\{\xi_{v_3}(i) : i \in B\}$. For $i \in B$ the path $Q_i(u_0)$ is a subgraph of the η-torso at u_0. It follows that $J \bigcup_{i\in B} Q_i(u_0) \bigcup_{i\in A} Z_i(u_0) \bigcup_{i\in A} W_i(u_0)$ is a linkage from $X_{\eta(v_1)}$ to $X_{\eta(v_2)}$, and so by the minimality of the specified u_0-linkages it is equal to the specified left u_0-linkage. It follows that u_0 has property B in η. $\square$

5.3 Reduced properties

Similarly to the 2-connected case, we have the following result:

Lemma 5.3.1. *Let (T, X) be a tree-decomposition of a graph G, let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) with orderings ξ_t, specified linkages and common intersection set I, let $\gamma : T_{h'} \hookrightarrow T_h$ be a monotone homeomorphic embedding, and let $\eta' := \eta \circ \gamma : T_{h'} \hookrightarrow T$ be a subcascade of η of height h'. Then for every major vertex $t_0 \in V(T_{h'})$*

(i) *η' is an ordered cascade with orderings $\xi_{\gamma(t)}$ and common intersection set I,*

(ii) *if the vertex $\gamma(t_0)$ has property A^m_{ijkl} (or B_{ijkl}, B^1_{ijkl}, resp.) in η, then t_0 has property A^m_{ijkl} (or B_{ijkl},B^1_{ijkl}, resp.) in η'.*

Furthermore, the specified linkages for η' may be chosen in such a way that

(iii) $(A_{t_0}, B_{t_0}, C_{t_0}, D_{t_0}) = (A_{\gamma(t_0)}, B_{\gamma(t_0)}, C_{\gamma(t_0)}, D_{\gamma(t_0)})$,

(iv) *the vertex t_0 has property B in η' if and only if $\gamma(t_0)$ has property B in η, and*

(v) *if the specified linkages for η are minimal, then the specified linkages for η' are minimal.*

Lemma 5.3.2. *There exists a positive integer h such that the following holds. Let $s \geq 4$ be an integer and let (T, X) be a tree-decomposition of a graph G. Let $\eta : T_h \hookrightarrow T$ be an ordered cascade in (T, X) of height h and size $|I| + s$ with orderings ξ_t and common* *intersection set I such that there exist some distinct $i, j, k, l \in \{1, 2, \dots, s\}$ and $m \in \{0, 1, 2, 3, 4\}$ such that every major vertex of T_h has property A^m_{ijkl}. Then there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one such that the unique major vertex of T_1 has property A^m_{ijkl} with ordered feet in η' if $m \in \{0, 1\}$ or T_1 has property A^m_{ijkl} with ordered left-feet in η' if $m \in \{2, 3, 4\}$.*

Proof. Let $h(a, k)$ be the function of Lemma 3.3.2. Let $h = h(4, (4!)^2)$. Assume u is an arbitrary major vertex of T_h and its trinity is (v_1, v_2, v_3). Assume the feet of L_i, L_j, L_k, L_l in $X_{\eta(v_2)}$ are x_1, x_2, x_3, x_4 and the feet of L_i, L_j, L_k, L_l in $X_{\eta(v_3)}$ are x_5, x_6, x_7, x_8. Then for every major vertex u of T_h, consider the tuple $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ as its color. By Lemma 3.3.2, there exists a monotone homeomorphic embedding $\gamma : T_4 \hookrightarrow T_h$ such that $\gamma(t)$ has the same tuple of eight feet for every major vertex $t \in V(T_4)$. Let $\eta_1 = \eta \circ \gamma : T_4 \hookrightarrow T$. By Lemma 5.3.1, η_1 is still an ordered cascade where every major vertex $t \in V(T_4)$ has property A^m_{ijkl}. Also, t has the same tuple of eight feet for every major vertex $t \in V(T_4)$.

Assume t_0 is a major root of T_1 and its trinity is (t_1, t_2, t_3). Let u_0 be the major root of T_4 and its trinity be (v, v_1, w_1). Let u be a major vertex in T_4 and let (v_1, v_2, v_3) be its trinity.

First assume $m \in \{2, 3, 4\}$. Let x_i, x_j, x_k, x_l be the end of $L_i(u)$ and the feet of $L_j(u), L_k(u), L_l(u)$ in $X_{\eta_1(v_1)}$, respectively. Let f be the function such that $f(x_h)$ are the feet of $L_h(u)$ in $X_{\eta_1(v_2)}$ for all $h \in \{j, k, l\}$ and f_{x_i} is the other end of $L_i(u)$. Define $f_0(x) = f(x)$ and $f_n(x) = f(f_{n-1}(x))$ for $n \geq 1$.

Let $\gamma_1(t_0) = u_0, \gamma_1(t_1) = v$, and $\gamma_1(t_3) = w_1$. For $h \in \{1, 2\}$, let u_h be the child of v_h and v_{h+1} be the left child of u_h. Let x_i, x_j, x_k, x_l be the feet of $L_i(u_0), L_j(u_0), L_k(u_0), L_l(u_0)$ in $X_{\eta(v)}$. Then there exists $h_1 \in \{1, 2, 3\}$ such that $f_{h_1}(x) = x$ for all $x \in \{x_i, x_j, x_k, x_l\}$. Let $\gamma_1(t_2) = v_{h_1}$ and $\eta' = \eta_1 \circ \gamma_1$. For $h \in \{i, j, k, l\}$, let

$$L_h = L_h(u_0) \cup \big(\bigcup_{1 \leq n < h_1} f_n(x_h) L_h(u_n) f_{n+1}(x_h) \big)$$

. Then these paths and tripods and $R_i(u_0)$ show that η' is as desired.

Therefore assume $m \in \{0, 1\}$. Let x_i, x_j, x_k, x_l be the feet of $L_i(u), L_j(u), L_k(u), L_l(u)$ in $X_{\eta_1(v_1)}$, respectively. Let f, g be functions such that $f(x_h)$ are the feet of $L_h(u)$ in $X_{\eta_1(v_2)}$ and $g(x_h)$ are the feet of $L_h(u)$ in $X_{\eta_1(v_3)}$ for all $h \in \{i, j, k, l\}$. Define $f_0(x) = f(x)$ and $f_n(x) = f(f_{n-1}(x))$ for $n \geq 1$, and $g_0(x) = g(x)$ and $g_n(x) = g(g_{n-1}(x))$ for $n \geq 1$.

Assume t_0 is a major root of T_1 and its trinity is (t_1, t_2, t_3). Let u_0 be the major root of T_4 and its trinity be (v, v_1, w_1). Let $\gamma_1(t_0) = u_0$ and $\gamma_1(t_1) = v$. For $h \in \{1, 2\}$, let u_h be the child of v_h and v_{h+1} be the left child of u_h, and let r_h be the child of w_h and w_{h+1} be the right child of r_h. Let x_i, x_j, x_k, x_l be the feet of $L_i(u_0), L_j(u_0), L_k(u_0), L_l(u_0)$ in $X_{\eta(v)}$. Then there exists $h_1, h_2 \in \{1, 2, 3\}$ such that $f_{h_1}(x) = x$ and $g_{h_2}(x) = x$ for all $x \in \{x_i, x_j, x_k, x_l\}$. Let $\gamma_1(t_2) = v_{h_1}$,$\gamma_1(t_3) = w_{h_2}$, and $\eta' = \eta_1 \circ \gamma_1$. For $h \in \{i, j, k, l\}$, let

$$L_h = L_h(u_0) \cup \Big(\bigcup_{1 \leq n < h_1} f_n(x_h) L_h(u_n) f_{n+1}(x_h) \Big) \cup \Big(\bigcup_{1 \leq n < h_2} g_n(x_h) L_h(r_n) g_{n+1}(x_h) \Big),$$

Then these tripods show that η' is as desired. $\square$

Lemma 5.3.3. *Let $s \geq 4$ be an integer and let (T, X) be a tree-decomposition of a graph G satisfying (W6). Let $\eta : T_3 \hookrightarrow T$ be an ordered cascade in (T, X) of height two and size $|I| + s$ with orderings ξ_t and common intersection set I such that there exist distinct $i, j, k, l \in \{1, 2, \ldots, s\}$ such that*

- *every major vertex of T_3 has property A^2_{ijkl} with ordered left-feet, or*
- *every major vertex of T_3 has property A^3_{ijkl} with ordered left-feet, or*
- *every major vertex of T_3 has property A^4_{ijkl} with ordered left-feet.*

Then there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one such that the unique major vertex of T_1 has property $A^1_{i'j'k'l'}$ in η', where $(i', j', k', l') = (i, j, k, l)$ for the first and third cases and $(i', j', k', l') = (j, i, k, l)$ for the second case.

Proof. Assume that the major root of T_3 is u_0 and its trinity is (v_1, v_2, v_3). Let u_1 be the major vertex at height one that is adjacent to v_2 and let its trinity be (v_2, v_4, v_5). Let the major root of T_1 be t_0 and its trinity be (t_1, t_2, t_3). Let x_h be the foot of $L_h(u_1)$ in $X_{\eta(v_5)}$. Let $\gamma(t_0) = u_0$, $\gamma(t_1) = v_1$, $\gamma(t_2) = v_5$, $\gamma(t_3) = v_3$. Then $\eta' = \eta \circ \gamma$ is as desired.

In fact, assume every major vertex of T_3 has property A^2_{ijkl} with ordered left-feet. For $h \in \{j, k, l\}$, let $L_h = L_h(u_0) \cup \xi_{v_2}(h)L_h(u_1)x_h$. Let $L_i = L_i(u_0) \cup L_i(u_1) \cup P \cup \xi_{v_2}(l)L_l(u_1)\xi_{v_4}(l) \cup R_i(u_1) \cup c_i(u_0)L_l(u_0)\xi_{v_2}(l) \cup R_i(u_0)$, where P is a W6-path in the outer graph at v_4 joining $\xi_{v_4}(i)$ and $\xi_{v_4}(l)$. Then these tripods show that t_0 has property A^1_{ijkl} in η'.

Assume every major vertex of T_3 has property A^3_{ijkl} with ordered left-feet. For $h \in \{j, k, l\}$, let $L_h = L_h(u_0) \cup \xi_{v_2}(h)L_h(u_1)x_h$. Let $L_i = L_i(u_0) \cup L_i(u_1) \cup P \cup \xi_{v_2}(k)L_k(u_1)\xi_{v_4}(k) \cup R_i(u_1) \cup c_i(u_0)L_k(u_0)\xi_{v_2}(k) \cup R_i(u_0)$, where P is a W6-path in the outer graph at v_4 joining $\xi_{v_4}(i)$ and $\xi_{v_4}(k)$. Then these tripods show that t_0 has property A^1_{jikl} in η'.

Therefore assume every major vertex of T_3 has property A^4_{ijkl} with ordered left-feet. Let $L_l = L_l(u_0) \cup \xi_{v_2}(l)L_l(u_1)x_l$. Assume y_h is the foot of $L_h(u_0)$ in $X_{\eta(v_3)}$ for all $h \in \{i, j, k\}$. Let z_1 be the vertex on $L_j(u_0) \cap L_k(u_0)$ that is closest to y_j. Then let $L_k = \xi_{v_1}(k)L_k(u_0)\xi_{v_2}(k) \cup \xi_{v_2}(k)L_k(u_1)x_k \cup z_1L_j(u_0)y_j$. Let z_2 be the vertex on $L_j(u_1) \cap L_k(u_1)$ that is closest to $c_j(u_1)$. Then let $L_j = \xi_{v_1}(j)L_j(u_0)\xi_{v_2}(j) \cup \xi_{v_2}(j)L_j(u_1)x_j \cup R_i(u_0) \cup c_iL_k(u_0)\xi_{v_2}(k) \cup \xi_{v_2}(k)L_k(u_1)z_2$. Let z_3 be the vertex on $L_k(u_0) \cap L_l(u_0)$ that is closest to y_k. Then let $L_i = \xi_{v_1}(i)L_i(u_0)\xi_{v_2}(i) \cup \xi_{v_2}(i)L_i(u_1)x_i \cup P_1 \cup \xi_{v_4}(k)L_k(u_1)c_i(u_1) \cup R_i(u_1) \cup y_kL_k(u_0)z_3 \cup z_3L_l(u_0)\xi_{v_2}(l) \cup \xi_{v_2}(l)L_l(u_1)\xi_{v_4}(l) \cup P_2$, where P_1 is a W6-path in the outer graph at v_4 joining $\xi_{v_4}(i)$ and $\xi_{v_4}(k)$ and P_2 is a W6-path in the outer graph at v_4 joining P_1 and $\xi_{v_4}(l)$. Then these tripods show that t_0 has property A^1_{ijkl} in η'. $\square$

Lemma 5.3.4. *For every integer $s \geq 4$ there exists a positive integer $h = h(s)$ such that the following holds. Let (T, X) be a tree-decomposition of a graph G satisfying (W6) and (W7). Let $\eta : T_h \hookrightarrow T$ be a regular cascade in (T, X) of height h and size $|I| + s$ with orderings ξ_t and common intersection set I such that every major vertex of T_h has property*

B. Then there exists a weak subcascade $\eta' : T_1 \hookrightarrow T$ of η of height one and distinct $i, j, k, l \in \{1, 2, ..., s\}$ such that the unique major vertex of T_1 has property B_{ijkl} in η'.

Proof. Let h be as in Lemma 3.3.2 applied to $a = 4$ and $k = (s/2)^2 + 2(s/2)^3$. Let the common confinement sets for η be A, B, C, D. Let the major root of T_1 be t_0 and its trinity be (t_1, t_2, t_3). Let the major root of T_4 be u_0 and its trinity be (w_1, w_2, w_3). Let two major vertices at height one of T_4 be u_1 and u_2. Assume the trinity at u_1 is (w_2, w_4, w_5) and the trinity at u_2 is (w_3, w_6, w_7). Let the child of w_6 be u_3 and the child of w_7 be u_4.

For a major vertex $w \in V(T_h)$ with trinity (v_1, v_2, v_3) there are disjoint paths in the η-torso at w as in the definition of property B. For $a \in A$ and $b \in B$ let $R_a(w)$ denote the path with ends $\xi_{v_1}(a)$ and $\xi_{v_2}(a)$, let $R_b(w)$ denote the path with ends $\xi_{v_1}(b)$ and $\xi_{v_3}(b)$, let $R_{ab}(w)$ denote the path with ends $\xi_{v_2}(b)$ and $\xi_{v_3}(a)$, let $Y_a(w)$ denote the path with ends $\xi_{v_2}(a)$ and $\xi_{v_2}(g(a))$, and let $Z_a(w)$ denote the path with ends $\xi_{v_3}(a)$ and $\xi_{v_3}(g(a))$, where g is the bijective function between A and B as in the definition of property B. Let g_h be the functions as in the definition of property B at major vertex u_h for all $h \in \{0, 1, 2, 3, 4\}$.

Let I be the common intersection set of η. Then $\eta(v_1), \eta(v_2), \eta(v_3)$ is a triad in T with center $\eta(w)$ and for all $i \in \{1, 2, 3\}$ we have $X_{\eta(v_i)} \cap X_{\eta(w)} = I = X_{\eta(v_1)} \cap X_{\eta(v_2)} \cap X_{\eta(v_3)}$, and hence the triad is not X-separable. By (W7) there is a path $R(w)$ connecting two of the three sets of disjoint paths in the η-torso at w.

If $R(w)$ goes from $R_a(w)$ to $R_b(w)$ for $a \in A$ and $b \in B$, we say it w has color (a, b). If $R(w)$ goes from $R_a(w)$ to R_{cb} for $a \in A$ or $a \in B$ and $b \in B, c \in A$, we say w has color (a, cb). By Lemma 3.3.2, there exists a monotone homeomorphic embedding $\gamma : T_4 \hookrightarrow T_h$ and $a \in A, b \in B$ such that $\gamma(t)$ has color (a, b) in η for every major vertex $t \in V(T_4)$, or there exists a monotone homeomorphic embedding $\gamma : T_4 \hookrightarrow T_h$ and $a \in A$ or $a \in B$ and $b \in B, c \in A$ such that $\gamma(t)$ has color (a, cb) in η for every major vertex $t \in V(T_4)$.

Assume there exists a monotone homeomorphic embedding $\gamma : T_4 \hookrightarrow T_h$ and $a \in A, b \in B$ such that $\gamma(t)$ has color (a, b) in η for every major vertex $t \in V(T_4)$. Let $\eta_1 = \eta \circ \gamma$, then by Lemma 5.3.1, t has property B in η_1 for every major vertex $t \in V(T_4)$

and one end of $R(t)$ is in the path $R_a(t)$ and the other end is in $R_b(t)$. Let $\gamma_1(t_0) = u_0, \gamma_1(t_1) = w_1, \gamma_1(t_2) = w_2$, and $\gamma_1(t_3) = w_7$. Let $\eta' = \eta_1 \circ \gamma_1$. Let $x_1 \in A$ be such that $g_0(x_1) = b$. If $x_1 = a$ then t_0 has property B_{abcd} in η' for some $c \in A - \{a\}$ and $d = g_0(c)$. Therefore assume $x_1 \neq a$. Let $d = g_0(a), x_2 = g_2(a)$ and c such that $g_2(c) = b$. Then argue similarly we also have $a \neq c$.

Claim 5.3.4.1. *Let w be a minor vertex of T_4 of height at most two. Then in the outer graph at w there are two disjoint paths from $\xi_w(a)$ to $\xi_w(b)$ and from $\xi_w(a')$ to $\xi_w(b')$ for any $a' \in A - \{a\}$ and $b' \in B - \{b\}$.*

In fact, let u be the child of w and let the trinity at u be (w, w', w''). Let $M_1 = \xi_w(a)R_a(u)x \cup R(u) \cup yR_b(w)\xi_w(b)$, where x and y are the ends of $R(u)$, and $M_2 = R_{a'}(u) \cup P_1 \cup R_{b''b}(u) \cup P_2 \cup R_{b'}(u)$, where b'' has b as the image in the function in the definition of property B at u, P_1 is a W6-path in the outer graph at w' between $\xi_{w'}(a')$ and $\xi_{w'}(b)$ and P_2 is a W6-path in the outer graph at w'' between $\xi_{w''}(b'')$ and $\xi_{w''}(b')$. Then M_1 and M_2 are two disjoint paths needed.

Denote the two paths in Claim 5.3.4.1 as $M_{ab}(u)$ and $M_{a'b'}(u)$. Back to the lemma, let $R_h = R_h(u_0)$ for all $h \in \{a, c\}$ and $R_h = R_h(u_0) \cup R_h(u_2)$ for all $h \in \{b, d\}$. Let $R_{ab} = R_{x_1 b}(u_0) \cup R_{x_1}(u_2) \cup M_{x_1 x_2}(u_3) \cup R_{a x_2}(u_2)$ and $R_{cd} = R_{ad}(u_0) \cup R_a(u_2) \cup M_{ab}(u_3) \cup R_{cb}(u_2)$. Let $Y_a = M_{ab}(u_1)$, $Y_c = M_{cd}(u_1)$, $Z_a = M_{ab}(u_4)$, and $Z_c = M_{cd}(u_4)$. Then these paths show that t_0 has property B_{abcd} in η'.

Therefore we can assume there exists a monotone homeomorphic embedding $\gamma : T_4 \hookrightarrow T_h$ and $a \in A$ or $a \in B$ and $b \in B, c \in A$ such that $\gamma(t)$ has color (a, cb) in η for every major vertex $t \in V(T_4)$. Without loss of generality, assume $a \in A$. Let $\eta_1 = \eta \circ \gamma$, then by Lemma 5.3.1, t has property B in η_1 for every major vertex $t \in V(T_4)$ and one end of $R(t)$ is in the path $R_a(t)$ and the other end is in $R_{cb}(t)$.

Claim 5.3.4.2. *Let w be a minor vertex of T_4 of height at most two. Then in the outer graph at w there are two disjoint paths from $\xi_w(a)$ to $\xi_w(b)$ and from $\xi_w(a')$ to $\xi_w(b')$ for* *any* $a' \in A - \{a\}$ *and* $b' \in B - \{b\}$.

In fact, let u be the child of w and let the trinity at u be (w, w', w''). Let $N_1 = \xi_w(a)R_a(u)x \cup R(u) \cup yR_{cb}(w)\xi_{w''}(c) \cup Z_c(u) \cup R_b(u)$, where x and y are the ends of $R(u)$. Let $N_2 = R_{a'}(u) \cup P \cup R_{b''b'}(u) \cup Z_{b''} \cup R_{b'}(u)$, where b'' has b' as the image in the function in the definition of property B at u and P is a W6-path in the outer graph at w' between $\xi_{w'}(a')$ and $\xi_{w'}(b')$. Then N_1 and N_2 are two disjoint paths needed.

Now denote the two paths in Claim 5.3.4.2 as $N_{ab}(u)$ and $N_{a'b'}(u)$. Back to the lemma, Let $\gamma_1(t_0) = u_0, \gamma_1(t_1) = w_1, \gamma_1(t_2) = w_2$, and $\gamma_1(t_3) = w_7$. Let $\eta' = \eta_1 \circ \gamma_1$. If $c = a$ then t_0 has property B_{abed} in η' for some $e \in A - \{a\}$ and $d = g_0(e)$. Therefore assume $c \neq a$. Let $d = g_0(a)$ and $x_1 = g_2(a)$. Let $R_h = R_h(u_0)$ for all $h \in \{a, c\}$ and $R_h = R_h(u_0) \cup R_h(u_2)$ for all $h \in \{b, d\}$. Let $R_{ab} = R_{cb}(u_0) \cup R_c(u_2) \cup P \cup R_{ax_1}(u_2)$, where P is a W6-path in the outer graph at w_6 between $\xi_{w_6}(c)$ and $\xi_{w_6}(x_1)$, and $R_{cd} = R_{ad}(u_0) \cup \xi_{w_3}(a)R_a(u_2)x \cup R(u_3) \cup yR_{cb}(u_2)\xi_{w_7}(c)$, where x and y are the ends of $R(u_3)$. Let $Y_a = N_{ab}(u_1)$, $Y_c = N_{cd}(u_1)$, $Z_a = N_{ab}(u_4)$, and $Z_c = N_{cd}(u_4)$. Then these paths show that t_0 has property B_{abcd} in η'. □

Lemma 5.3.5. *Let* $s \geq 4$ *be an integer and let* (T, X) *be a tree-decomposition of a graph* G *satisfying (W6). Let* $\eta : T_3 \hookrightarrow T$ *be a regular cascade in* (T, X) *of size* $|I|+s$ *with orderings* ξ_t *and common intersection set* I *such that there exist distinct* $i, j, k, l \in \{1, 2, ..., s\}$ *such that every major vertex of* T_3 *has property* B_{ijkl}. *Then there exists a weak subcascade* $\eta' : T_1 \hookrightarrow T$ *of* η *of height one such that the unique major vertex of* T_1 *has property* B^1_{ijkl} *in* η'.

Proof. Assume that three major vertices at height zero and one of T_3 are u_0, u_1, u_2. Let the trinity at u_0 be (v_1, v_2, v_3), the trinity at u_1 be (v_2, v_4, v_5), and the trinity at u_2 be (v_3, v_6, v_7). Assume the major vertex of T_1 is t_0, and its trinity is (t_1, t_2, t_3). For a major vertex $w \in V(T_3)$ let $R_i(t_0), R_j(t_0), R_k(t_0), R_l(t_0), R_{ij}(t_0), R_{kl}(t_0), R(t_0), Y_i(t_0)$, $Y_k(t_0), Z_i(t_0), Z_k(t_0)$ be as in the definition of property B_{ijkl}.

We need to find a weakly monotone homeomorphic embedding $\gamma : T_1 \hookrightarrow T_3$ such that $\eta' = \eta \circ \gamma$ satisfies the requirement. Set $\gamma(t_0) = u_0$ and $\gamma(t_1) = v_1$. Our choice for $\gamma(t_2)$ will be v_4 or v_5, depending on which two of the three paths $R_i(u_1), R_j(u_1), R_{ij}(u_1)$ in the η-torso at u_1 the path $R(u_1)$ is connecting. If $R(u_1)$ is between $R_i(u_1)$ and $R_j(u_1)$, then choose either v_4 or v_5 for $\gamma(t_2)$. If $R(u_1)$ is between $R_i(u_1)$ and $R_{ij}(u_1)$, then set $\gamma(t_2) = v_4$, and if it is between $R_j(u_1)$ and $R_{ij}(u_1)$, then set $\gamma(t_2) = v_5$. Do this similarly for $\gamma(t_3)$. Then $\eta' = \eta \circ \gamma$ will satisfy the requirement. In fact, we will prove this for the case when $R(u_1)$ is between $R_i(u_1)$ and $R_{ij}(u_1)$ and $R(u_2)$ is between $R_j(u_2)$ and $R_{ij}(u_2)$. The other cases are similar.

In this case, our choice is $\gamma(t_0) = u_0, \gamma(t_1) = v_1, \gamma(t_2) = v_4, \gamma(t_3) = v_7$. Assume the end of $R(u_1)$ in $R_{ij}(u_1)$ is x and the end of $R(u_2)$ in $R_{ij}(u_2)$ is y. Now let

$$P = xR_{ij}(u_1)\xi_{v_5}(i) \cup Z_i(u_1) \cup R_j(u_1) \cup R_{ij}(u_0) \cup R_i(u_2) \cup Y_i(u_2) \cup \xi_{v_6}(j)R_{ij}(u_2)y,$$

$$L_i = R_i(u_0) \cup R_i(u_1) \cup R(u_1) \cup P \cup yR_{ij}(u_2)\xi_{v_7}(i),$$

$$L_j = R_j(u_0) \cup R_j(u_2) \cup R(u_2) \cup P \cup xR_{ij}(u_1)\xi_{v_4}(j),$$

$$R_k = R_k(u_0) \cup R_k(u_1),$$

$$R_l = R_l(u_0) \cup R_l(u_1),$$

and

$$R_{kl} = R_{kl}(u_1) \cup Z_k(u_1) \cup R_l(u_1) \cup R_{kl}(u_0) \cup R_k(u_2) \cup Y_k(u_2) \cup R_{kl}(u_2).$$

Then the paths and tripods show that the major vertex of $\eta' = \eta \circ \gamma : T_1 \hookrightarrow T$ has property B^1_{ijkl}. $\square$

Lemma 5.3.6. *For every positive integers h' and $w \geq 4$ there exists a positive integer $h = h(h', w)$ such that the following holds. Let s be a positive integer such that $4 \leq s \leq w$.*

Let (T, X) be a tree-decomposition of a graph G of width less than w and satisfying (W6)-(W7). Assume there exists a regular cascade $\eta : T_h \hookrightarrow T$ of size $|I| + s$ with specified linkages that are minimal, where I is its common intersection set. Then there exist distinct integers $i, j, k, l \in \{1, 2, \ldots, s\}$ and a weak subcascade $\eta' : T_{h'} \hookrightarrow T$ of η of height h' such that

- *every major vertex of $T_{h'}$ has property A^0_{ijkl} with ordered feet in η', or*
- *every major vertex of $T_{h'}$ has property A^{1a}_{ijkl} with ordered feet in η', or*
- *every major vertex of $T_{h'}$ has property B^1_{ijkl} in η'*

Proof. Let $h(a, k)$ be the function of Lemma 3.3.2. Let h_1 be the h in Lemma 5.3.2 and h_2 be the h in Lemma 5.3.4. Let $a_5 = h(h', 2), a_4 = \max\{3a_5, h(h', 2)\}$, $a_3 = \max\{3a_4, h_1 a_4\}$, $a_2 = h(a_3, 5.4!\binom{w}{4})$, $a_1 = \max\{7a_2, h_2 a_2\}$ and $h = h(a_1, 2)$. Consider having property B or not having property B as colors, then by Lemma 3.3.2 there exists a monotone homeomorphic embedding $\gamma : T_{a_1} \hookrightarrow T_h$ such that either $\gamma(t)$ has property B in η for every major vertex $t \in V(T_{a_1})$ or $\gamma(t)$ does not have property B in η for every major vertex $t \in V(T_{a_1})$. By Lemma 5.3.1 $\eta_1 = \eta \circ \gamma : T_{a_1} \hookrightarrow T$ is still a regular cascade with specified linkages that are minimal. Also, either t has property B in η_1 for every major vertex $t \in V(T_{a_1})$ or t does not have property B in η_1 for every major vertex $t \in V(T_{a_1})$.

If t has property B in η_1 for every major vertex $t \in V(T_{a_1})$, then by Lemma 5.3.4 there exists a weak subcascade η_2 of η_1 of height a_2 such that every major vertex of T_{a_2} has property B_{ijkl} in η_2 for some distinct $i, j, k, l \in \{1, 2, ..., s\}$. Consider each choice of tuple (i, j, k, l) as a color; then by Lemma 3.3.2 there exists a monotone homeomorphic embedding $\gamma_1 : T_{a_3} \hookrightarrow T_{a_2}$ such that for some distinct $i, j, k, l \in \{1, 2, ..., s\}$, $\gamma_1(t)$ has property B_{ijkl} in η_2 for every major vertex $t \in V(T_{a_3})$. Let $\eta_3 = \eta_2 \circ \gamma_1$. Then by Lemma 5.3.1 this implies t has property B_{ijkl} in η_3 for every major vertex $t \in V(T_{a_3})$. By Lemma 5.3.5 there exists a weak subcascade η_4 of η_3 of height a_4 such that every major vertex of T_{a_4} has property B^1_{ijkl} in η_4. Hence η_4 is as desired.

If t does not have property B in η_1 for every major vertex $t \in V(T_{a_1})$, then by Lemma 5.2.1 there exists a weak subcascade η_2 of η_1 of height a_2 such that every major vertex of T_{a_2} has property A_{ijkl} for some distinct $i, j, k, l \in \{1, 2, ..., s\}$. By Lemma 5.1.1, every major vertex of T_{a_2} has property A^m_{ijkl} for some distinct $i, j, k, l \in \{1, 2, ..., s\}$ and $m \in \{0, 1, 2, 3, 4\}$. Consider each property A^m_{ijkl} as a color; then by Lemma 3.3.2 there exists a monotone homeomorphic embedding $\gamma_1 : T_{a_3} \hookrightarrow T_{a_2}$ such that for some distinct $i, j, k, l \in \{1, 2, ..., s\}$ and $m \in \{0, 1, 2, 3, 4\}$, $\gamma_1(t)$ has property A^m_{ijkl} in η_2 for every major vertex $t \in V(T_{a_3})$. Let $\eta_3 = \eta_2 \circ \gamma_1$, then t has property A^m_{ijkl} in η_3 for every major vertex $t \in V(T_{a_3})$ by Lemma 5.3.1.

If $m \in \{0, 1\}$, by Lemma 5.3.2, there exists a weak subcascade η_4 of η_3 of height a_4 such that every major vertex of T_{a_4} has property A^m_{ijkl} with ordered feet. If $m = 0$ then η_4 is as desired. If $m = 1$, then by Lemma 3.3.2 there exists a monotone homeomorphic embedding $\gamma_2 : T_{h'} \hookrightarrow T_{a_4}$ such that either $\gamma_2(t)$ has property A^{1a}_{ijkl} in η_4 for every major vertex $t \in V(T_{h'})$ or $\gamma_2(t)$ has property A^{1b}_{ijkl} in η_4 for every major vertex $t \in V(T_{h'})$. Let $\eta_5 = \eta_4 \circ \gamma_2$, then by Lemma 5.3.1 t has property A^{1a}_{ijkl} in η_5 for every major vertex $t \in V(T_{h'})$ or t has property A^{1b}_{ijkl} in η_5 for every major vertex $t \in V(T_{h'})$. If t has property A^{1b}_{ijkl} then it also has property B^1_{jkil}, so η_5 is as desired.

If $m \in \{2, 3, 4\}$, by Lemma 5.3.2, there exists a weak subcascade η_4 of η_3 of height a_4 such that every major vertex of T_{a_4} has property A^m_{ijkl} with ordered left-feet. By Lemma 5.3.3, there exists a weak subcascade η_5 of η_4 of height a_5 and distinct $i', j', k', l' \in \{1, 2, ..., s\}$ such that every major vertex of T_{a_5} has property $A^1_{i'j'k'l'}$ with ordered feet. By Lemma 3.3.2 there exists a monotone homeomorphic embedding $\gamma_2 : T_{h'} \hookrightarrow T_{a_5}$ such that either $\gamma_2(t)$ has property A^{1a}_{ijkl} in η_5 for every major vertex $t \in V(T_{h'})$ or $\gamma_2(t)$ has property A^{1b}_{ijkl} in η_5 for every major vertex $t \in V(T_{h'})$. Let $\eta_6 = \eta_5 \circ \gamma_2$, then by Lemma 5.3.1 t has property A^{1a}_{ijkl} in η_6 for every major vertex $t \in V(T_{h'})$ or t has property A^{1b}_{ijkl} in η_6 for every major vertex $t \in V(T_{h'})$. If t has property A^{1b}_{ijkl} then it also has property B^1_{jkil}, so η_6 is as desired. $\square$

5.4 Proof of Theorem 1.1.6

Lemma 5.4.1. *If a graph H has three distinct vertices u, v, w such that $H\backslash\{u, v, w\}$ is a forest, then there exists an integer n such that H is isomorphic to a minor of $\mathcal{P}''_n$.*

Proof. Let $u, v, w \in V(H)$ be such that $T := H\backslash\{u, v, w\}$ is a forest. We may assume, by replacing H by a graph with an H minor, that T is isomorphic to CT_t for some t, and that each of u, v, w is adjacent to every vertex of T. It follows that H is isomorphic to a minor of $\mathcal{P}_{2t}$, as desired. $\square$

Lemma 5.4.2. *Let H be a graph with two distinct vertices u, v such that $H\backslash\{u, v\}$ is an outerplanar graph. Then there exists an integer n such that H is isomorphic to a minor of $\mathcal{Q}''_n$.*

Proof. By Lemma 3.1.4, there exists an integer t such that $H\backslash\{u, v\}$ is isomorphic to a minor of $\mathcal{Q}_t$. We may assume, by replacing H by a graph with an H minor, that $H\backslash\{u, v\}$ is isomorphic to $\mathcal{Q}_t$ for some t, and that each of u, v is adjacent to every vertex of $\mathcal{Q}_t$. It follows that H is isomorphic to a minor of $\mathcal{Q}''_{t+2}$. $\square$

Lemma 5.4.3. *Let H be a tree plus a cycle going through its leaves in order from the leftmost leaf to the rightmost leaf and a vertex v adjacent to the leaves of the tree. Then there exists an integer n such that H is isomorphic to a minor of $\mathcal{R}''_n$.*

Proof. Let T be the tree in $H\backslash\{v\}$ and C be the cycle going through its leaves. We may assume, by replacing $H\backslash\{v\}$ by a graph with an $H\backslash\{v\}$ minor, that T is isomorphic to CT_t for some t, that C goes through its leaves in order from the leftmost leaf to the rightmost leaf, and that v is adjacent to every leaf of T. It follows that H is isomorphic to a minor of $\mathcal{R}''_t$, as desired. $\square$

Lemma 5.4.4. *Let H be a planar graph that consists of an outerplanar graph with a cycle going through its degree-2 vertices. Then there exists an integer n such that H is isomorphic to a minor of $\mathcal{S}''_n$.*

Proof. Let Q be the outerplanar graph and C be the cycle going through its degree-2 vertices in H. By Lemma 3.1.4, there exists an integer t such that Q is isomorphic to a minor of $\mathcal{Q}_t$. We may assume, by replacing H by a planar graph with an H minor, that Q is isomorphic to $\mathcal{Q}_t$ for some t, and that C goes through the leaves of $\mathcal{Q}_t$. It follows that H is isomorphic to $\mathcal{S}''_t$. □

By Lemmas 5.4.1, 5.4.4, 5.4.3 and 5.4.2 Theorem 1.1.6 is equivalent to the following theorem.

Theorem 5.4.5. *For every positive integer n, there exists an integer $p = p(n)$ such that every 4-connected graph with path-width at least p has $\mathcal{P}''_n$, $\mathcal{Q}''_n$, $\mathcal{R}''_n$ or $\mathcal{S}''_n$ as a minor.*

We deduce Theorem 5.4.5 from the following lemma.

Lemma 5.4.6. *Let n and w be positive integers. There exists an integer $p = p(n, w)$ such that for every 4-connected graph G, if G has tree-width less than w and path-width at least p, then G has a minor isomorphic to $\mathcal{P}''_n$, $\mathcal{Q}''_n$, $\mathcal{R}''_n$ or $\mathcal{S}''_n$.*

Proof. Let h_1 be as in Lemma 3.5.3 applied to $k = n$ and w. Let h_2 be as in Lemma 4.4.5. Let h_3 be the number as in Lemma 5.3.6 applied to $4n+1$ and w. Let $h = \max\{h_1, h_2, h_3, 2n+1\}$. Let p be as in Theorem 3.3.5 applied to $a = h$ and w. By Theorem 3.3.5, there exists a tree-decomposition (T, X) of G such that:

- (T, X) has width less than w,
- (T, X) satisfies (W1)–(W7), and
- for some s, where $4 \leq s \leq w$, there exists a regular cascade $\eta : T_h \hookrightarrow T$ of height h and size s in (T, X) with specified t_0-linkages that are minimal for every major vertex $t_0 \in V(T_h)$.

Let I be the common intersection set of η, let ξ_t be the orderings, and let $s_1 = s - |I|$. Then $s_1 \geq 1$ by the definition of injective cascade.

Assume that $s_1 = 1$. Since $s \geq 4$, it follows that $|I| \geq 3$. Let $x, y, z \in I$. Let R be the union of the left and right specified t-linkage with respect to η, over all major vertices $t \in V(T_h)$ at height at most $h-2$. The minimality of the specified linkages implies that R is isomorphic to a subdivision of T_{h-1}. Let t be a minor vertex of T_h at height $h-1$. By Lemma 3.4.2 there exist three W6-paths with one end $\xi_t(1)$ and the other end x, y, or z in the outer graph at t. The union of R and these W6-paths shows that G has a $\mathcal{P}_n''$ minor, as desired.

Assume that $s_1 = 2$. Since $s \geq 4$, it follows that $|I| \geq 2$. By Lemma 3.5.3(iii), G has a $\mathcal{P}_n''$ minor or a $\mathcal{Q}_n''$ minor, as desired.

Assume that $s_1 = 3$. Since $s \geq 4$, it follows that $I \neq \emptyset$. By Lemma 4.4.5(ii), G has a $\mathcal{P}_n''$ or $\mathcal{R}_n''$ minor, as desired.

We may therefore assume that $s_1 \geq 4$. Let $h' = 4n + 1$. By Lemma 5.3.6 there exist distinct integers $i, j, k, l \in \{1, 2, \ldots, s\}$ and a weak subcascade $\eta' : T_{h'} \hookrightarrow T$ of η of height h' such that

- every major vertex of $T_{h'}$ has property A^0_{ijkl} with ordered feet in η', or
- every major vertex of $T_{h'}$ has property A^{1a}_{ijkl} with ordered feet in η'
- every major vertex of $T_{h'}$ has property B^1_{ijkl} in η'

Assume that every major vertex of $T_{h'}$ has property A^0_{ijkl} with ordered feet in η', and let R be the union of the corresponding tripods, over all major vertices $t \in V(T_{h'})$ at height at most $h'-2$. It follows that R is the union of four disjoint trees, each containing a subtree isomorphic to a subdivision of $T_{(h'-1)/2}$. Let t be a minor vertex of $T_{h'}$ at height $h'-1$. By Lemma 3.4.2 there exist W6-paths with ends $\xi_t(h)$ and $\xi_t(l)$ for all $h \in \{i, j, k\}$ in the outer graph at t. By contracting the tree that contains $\xi_t(h)$ for all $h \in \{i, j, k\}$, and by considering these W6-paths we deduce that G has a $\mathcal{P}_n''$ minor, as desired.

Assume next that every major vertex of $T_{h'}$ has property A^{1a}_{ijkl} with ordered feet in η'. Let the major root of $T_{h'}$ be u_0 and its left child be v. For every major vertex u that is a descendant of v, let $L_i(u), L_j(u), L_k(u), L_l(u)$ be the four tripods in the η'-torso at u as in the definition of property A^1_{ijkl}, and let $a(u), b(u)$ be the two ends of the path $L_i(u) \cap L_l(u)$. Let

$$R_1 = \bigcup_u (\xi_{v_1}(i) L_i(u) \xi_{v_2}(i) \cup \xi_{v_1}(l) L_l(u) \xi_{v_3}(l) \cup$$

$$\cup\, \xi_{v_2}(l) L_l(u) a(u) \cup a(u) L_l(u) b(u) \cup b(u) L_i(u) \xi_{v_3}(i)),$$

$$R_2 = \bigcup_u L_j(u),$$

and

$$R_3 = \bigcup_u L_k(u),$$

where the unions are taken over all major vertices u at height at most $h' - 2$ that are descendants of v and (v_1, v_2, v_3) here is the trinity at u. Then R_1, R_2, R_3 are disjoint and R_2 and R_3 is a tree isomorphic to a subdivision of $T_{(h'-1)/2}$. Let t be a minor vertex of $T_{h'}$ at height $h' - 1$. By Lemma 3.4.2 there exist W6-paths with ends $\xi_t(h)$ and $\xi_t(k)$ for all $h \in \{i, j, l\}$ in the outer graph at t. By Lemma 3.4.2, there exists a W6-path with ends $\xi_v(i)$ and $\xi_v(l)$ in the subgraph of G induced by $\bigcup X_r - I$, where the union is taken over all r in the component containing $\eta'(u_0)$ of $T - \eta'(v)$. By contracting R_2 to one vertex and considering that vertex and R_1, R_3 and these W6-paths we deduce that G has a $\mathcal{R}''_n$ minor, as desired.

We may therefore assume that every major vertex of $T_{h'}$ has property B^1_{ijkl} in η'. For every major vertex u in $T_{h'}$, let $L_i(u), L_j(u)$ and $R_k(u), R_l(u), R_{kl}(u)$ be as in the definition of property B_{ijkl}. Let the major root of $T_{h'}$ be u_0 and its left child be v. Let

$$R_1 = \bigcup_u \big(R_k(u) \cup R_l(u) \cup R_{kl}(u)\big) \text{ and } R_2 = \bigcup_u \big(L_i(u) \cup L_j(u)\big),$$

where the unions are taken over all major vertices u at height at most $h' - 2$ that are descendants of v. Then R_1 is disjoint from R_2, which contains a subgraph isomorphic to a subdivision of $\mathcal{Q}_{h'-3}$ by Lemma 3.5.2 applied to the graph R_2. By considering R_1, R_2 and the W6-paths as in the above case, we deduce that G has a $\mathcal{S}''_{h'-3}$ minor, as desired. $\square$

Proof of Theorem 5.4.5. Let a positive integer n be given. By Theorem 1.1.1 there exists an integer w such that every graph of tree-width at least w has a minor isomorphic to $\mathcal{S}''_n$. Let $p = p(n, w)$ be as in Lemma 5.4.6. We claim that p satisfies the conclusion of the theorem. Indeed, let G be a 4-connected graph of path-width at least p. By Theorem 1.1.1, if G has tree-width at least w, then G has a minor isomorphic to $\mathcal{S}''_n$, as desired. We may therefore assume that the tree-width of G is less than w. By Lemma 5.4.6 G has a minor isomorphic to $\mathcal{P}''_n, \mathcal{Q}''_n, \mathcal{R}''_n$ or $\mathcal{S}''_n$, as desired. $\square$

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